R–GROUPS, ELLIPTIC REPRESENTATIONS, AND PARAMETERS FOR $GSpin$ GROUPS

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Abstract. We study parabolically induced representations for $GSpin_m(F)$ with F a p–adic field of characteristic zero. The Knapp-Stein R–groups are described and shown to be elementary two groups. We show the associated cocycle is trivial proving multiplicity one for induced representations. We classify the elliptic tempered spectrum. For $GSpin_{2n+1}(F)$, we describe the Arthur (Endoscopic) R–group attached to Langlands parameters, and show these are isomorphic to the corresponding Knapp-Stein R–groups.

Introduction

We continue our study of parabolically induced representations for p–adic groups of classical type. Here we turn our attention to the group $GSpin_m(F)$, as defined by Asgari [4]. These are groups of type $B_{[m/2]}$ if m is odd and type $D_{m/2}$ if m is even. A long term goal is to study the group $Spin_m(F)$, which is the simply connected split group of type B or D, depending on whether m is odd or even, respectively. The advantage of studying $GSpin$ groups is their Levi subgroups are nicer, making the problem more tractable. We hope to apply the information derived here to $Spin$ groups, and we leave this to further study.

Let F be a nonarchimedean field of characteristic zero, and suppose $\mathbf{G}$ is a connected reductive quasi-split group defined over F. We denote the F–points, $\mathbf{G}(F)$, by G and use this notational convention throughout this manuscript. The admissible dual of G can be be studied through the theory of parabolically induced representations, as described in Harish-Chandra's philosophy of cusp forms [16]. Moreover, the discrete, tempered, and admissible spectra are classified through parabolic induction from supercuspidal, discrete series, and tempered representations (via the Langlands Classification) [30]. One also wishes to divide the tempered spectrum into the elliptic classes, [2], which are those which contribute to the Placherel Formula, and the non-elliptic classes. For this purpose, we let $\mathcal{E}_c(G), \mathcal{E}_t(G), \mathcal{E}_2(G)$, and $^\circ\mathcal{E}(G)$ be the classes of irreducible admissible, tempered, discrete series, and unitary supercuspidal representations, respectively, of G. We make no distinction between a representation π and its class $[\pi] \in \mathcal{E}_c(G)$.

Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$, and suppose $\sigma \in \mathcal{E}_2(M)$. We let $\mathrm{Ind}_P^G(\sigma)$ or $i_{G,M}(\sigma)$ denote the representation of G obtained through normalized induction from P, with σ extended trivially from M to P. In the case of archimedean groups, Knapp and Stein developed the theory of standard and normalized intertwining operators (see [21], for example). Through a combinatorial study of the inductive properties of these normalized intertwining operators they were able to describe a finite group, $R(\sigma)$, whose representation theory classifies the components of $i_{G,M}(\sigma)$, in that there is a bijection $\rho \mapsto \pi_\rho$ from the irreducible representations $\widehat{R(\sigma)}$ to the inequivalent components of $i_{G,M}(\sigma)$. More precisely, the intertwining algebra $\mathcal{C}(\sigma)$ of $i_{G,M}(\sigma)$ is isomorphic to the twisted group algebra $\mathbb{C}[R(\sigma)]_\eta$, with η a particular 2–cocylce of $R(\sigma)$ arising from composition of intertwining operators [2, 20]. In the archimedean case, $R(\sigma)$ is always abelian (in fact an elementary 2–group), so each ρ is a character and π_ρ appears in $i_{G,M}(\sigma)$ with multiplicity one. Silberger [28, 29] extended the theory of R-groups to p–adic fields. Knapp and Zuckerman [22] showed there are cases when $R(\sigma)$ would be non-abelian, and hence the multiplicity of π_ρ could be greater than one.

If $\mathbf{G} = \mathbf{G}_n = GSpin_{2n}$, or $GSpin_{2n+1}$, then any Levi subgroup is of the form

$$\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times \mathbf{G}_m,$$

with $n_1 + \cdots + n_r + m = n$. So, for any $\sigma \in \mathcal{E}_2(M)$ we have

$$\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \tau,$$

with $\sigma_i \in \mathcal{E}_2(GL_{n_i}(F))$, and $\tau \in \mathcal{E}_2(G_m)$. The similarity between this situation and that of the classical groups $Sp_{2n}(F)$ and $SO_n(F)$, makes it amenable to the techniques of [12]. In fact we prove the R–groups have the same structure as these classical groups. Thus, our first main results can be phrased as R-groups for $GSpin$ groups mirror those for split classical groups (cf. Theorems 2.5 and 2.7). In particular, $R(\sigma) \simeq \mathbb{Z}_2^d$, for some $0 \leq d \leq r$.

Arthur [2] undertook the study of the elliptic spectrum, and was able to use the extension of $R(\sigma)$ defined by η to characterize when components of $i_{G,M}(\sigma)$ have elliptic components. Herb [18] used this characterization, along with the description of the R–groups in [12], to determine the elliptic tempered spectrum of $Sp_{2n}(F)$ and $SO_n(F)$. Because the description of R–groups in our case is similar to that of [12], the techniques of [18] can be applied, and again the results are similar. To be more precise, the cocycle η always splits and $i_{G,M}(\sigma)$ has elliptic components if and only if d is as large as possible (this turns out to be $d = r$ or $r - 1$ cf. Lemma 3.1 and Theorems 3.3 and 3.4).

On the other hand, the local Langlands conjecture predicts a canonical bijection $\varphi \to \Pi_\varphi(G)$ between admissible homomorphisms $\varphi : W_F' \to {}^L G$ and L–packets $\Pi_\varphi(G)$ of G. Here, W_F' is the

Weil-Deligne group, ${}^LG = \hat{G} \rtimes W_F$ is the Langlands L–group, with $\hat{G}$ the connected Langlands dual group, and W_F is the Weil group. The L–packets $\Pi_\varphi(G)$ are finite sets which partition $\mathcal{E}_c(G)$, and the members of $\Pi_\varphi(G)$ are to be L–indistinguishable, in the sense that the Langlands L–functions and ε–factors are to be constant on $\Pi_\varphi(G)$. If $\sigma \in \mathcal{E}_2(M,)$ and $\varphi : W_F' \to {}^LM$ is its Langlands parameter (i.e., $\sigma \in \Pi_\varphi(M)$), then composing with the inclusion ${}^LM \hookrightarrow {}^LG$ gives an L-packet $\Pi_\varphi(G)$, and the elements of this L–packet should be all components of $i_{G,M}(\sigma')$, with $\sigma' \in \Pi_\varphi(M)$. Langlands predicted the R–group, $R(\sigma)$ should be encoded in this arithmetic information, and Arthur made this more precise in [1]. In particular, Arthur defined a finite group $R_{\varphi,\sigma}$ attached to φ and σ, and predicts $R(\sigma) \simeq R_{\varphi,\sigma}$. This conjecture has been confirmed in many cases [6, 7, 9, 10, 14, 20, 27].

Here we are able to prove $R(\sigma) \simeq R_{\varphi,\sigma}$ for $GSpin_{2n+1}$ in several steps. The first is to reduce the isomorphism to the case where $\mathbf{M}$ is maximal, and this we do in the wider context of split groups (cf. Lemma 4.1). Arthur identifies the stabilizer $W(\sigma)$ of σ in the Weyl group with a subgroup $W_{\varphi,\sigma}$ of a certain Weyl group in $\hat{M}$. $R(\sigma)$ can be realized as a quotient $W(\sigma)/W'$ of $W(\sigma)$, with W' the subgroup of $W(\sigma)$ generated by root reflections in the zeros of the rank 1 Plancherel measures. On the other hand $R_{\varphi,\sigma} = W_{\varphi,\sigma}/W^\circ_{\varphi,\sigma}$ is a quotient of $W_{\varphi,\sigma}$, where $W^\circ_{\varphi,\sigma}$ is the the intersection of $W_{\varphi,\sigma}$ with another, smaller Weyl group. Thus, it is enough to show, under the isomorphism of $W(\sigma)$ and $W_{\varphi,\sigma}$ that W' is identified with $W^\circ_{\varphi,\sigma}$. Hence it is enough to show $W^\circ_{\varphi,\sigma}$ is generated by co-root reflections coming from the roots for which the Plancherel measures are zero. Shahidi [26] showed, in the generic case, the zeros of the rank 1 Plancherel measures are equivalent to poles of Langlands L–functions, $L(s,\sigma,r_i)$, (where $i = 1, 2$ is determined in a particular way [25] and r_i is a representation of ${}^LM$ coming from its adjoint representation). The local Langlands conjecture predicts $L(s,\sigma,r_i) = L(s, r_i \circ \varphi)$, where the right hand side is the Artin L–function. We separate the proof of the isomorphism of Knapp-Stein and Arthur R–groups into two (maximal) cases, the Siegel parabolic subgroup, i.e $\mathbf{M} \simeq GL_n \times GL_1$, and the non-Siegel maximal parabolic subgroups, $\mathbf{M} \simeq GL_k \times \mathbf{G}_m$, with $m \geq 2$. The final results in these two cases can be found in Corollary 4.6 and Theorem 4.12. For the latter we need conjecture 9.4 of [26] (otherwise known as the Tempered L–packet Conjecture).

The structure and isomorphism of Knapp-Stein and Arthur R–groups plays a crucial role in the transfer of automorphic forms from classical to general linear groups in [3], and among the important results therein is a proof of the Tempered L–packet Conjecture in the case of classical groups. We expect if the methods of [3] can be extended to $GSpin$ groups, then the isomorphism of $R(\sigma)$ and $R_{\varphi,\sigma}$ would play a similar role.

In Section 1 we recall the basic facts about the $GSpin$ groups. In Section 2 we work to determine the zeros of the Plancherel measures and compute the R–groups for $GSpin$ groups. In Section 3 we show the cocycle which, along with the R–group, determines the structure of $i_{G,M}(\sigma)$ is a coboundary. We then use the results of Section 2 to classify the elliptic tempered spectra of $GSpin$ groups. In Section 4 we prove the isomorphism of the Knapp-Stein and Arthur R–groups for the $GSpin_{2n+1}$ groups.

1. Preliminaries

Let F be a local nonarchimedean field of characteristic zero. Let $\mathbf{G} = \mathbf{G}_n = GSpin_{2n}$, or $GSpin_{2n+1}$. We adopt the convention that $G_0 = GL_1$.We let $\mathbf{H} = Spin_{2n}$ or $Spin_{2n+1}$. We recall the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathbf{H} \to \mathbf{H}' \to 1,$$

where $H' = SO_{2n}$ or SO_{2n+1}, We have $\mathbf{G}$ and $\mathbf{H}$ are of type D_n in the first case and type B_n is the second case. Let $\hat{G}$ be the connected component of the Langlands L–group. Then $\hat{G} = GSO_{2n}(\mathbb{C})$ if $\mathbf{G} = GSpin_{2n}$ and is GSp_{2n} if $\mathbf{G} = GSpin_{2n+1}$. Then since $\mathbf{G}$ is split, ${}^LG = \hat{G} \times W_F$, with W_F the Weil group of F. We fix $\mathbf{B}$ to be the Borel subgroup in $\mathbf{G}$ lying over the upper triangular Borel subgroup in $\mathbf{H}'$. Let $\mathbf{B} = \mathbf{TU}$ be the Levi decomposition of $\mathbf{B}$. Let $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ be the roots of $\mathbf{T}$ in $\mathbf{G}$, and let Δ be the simple roots determined by $\mathbf{B}$. Then $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, where $\alpha_i = e_i - e_{i+1}$, for $i = 1, 2, \ldots, n-1$, and

$$\alpha_n = \begin{cases} e_{n-1} + e_n & \text{if } \mathbf{G} = GSpin_{2n}, \\ e_n & \text{if } \mathbf{G} = GSpin_{2n+1}. \end{cases}$$

Recall the Weyl group is $W = W(\mathbf{G}, \mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Note, if $\mathbf{G} = GSpin_{2n+1}$, then $W \simeq S_n \ltimes \mathbb{Z}_2^n$, while if $\mathbf{G} = GSpin_{2n}$, we have $W \simeq S_n \ltimes \mathbb{Z}_2^{n-1}$. One can compute this directly from the description in [5], or one can note that $W(\hat{G}, \hat{T})$ is of this form, and use duality. Taking this last approach, the description of these Weyl groups given in [13], which we summarize. Note

$$\hat{T} = \left\{ \operatorname{diag}\left\{ a_1, a_2, \cdots, a_n, \lambda a_n^{-1}, \ldots, \lambda a_2^{-1}, \lambda a_1^{-1} \right\} \middle| a_i, \lambda \in \mathbb{C}^\times \right\}$$

in either case. We may denote an element of $\hat{T}$ by $t(a_1, a_2, \ldots, a_n, \lambda)$. If $s \in S_n$, then we also denote by $\hat{s}$ a representative of the element of $W(\hat{G}, \hat{T})$ such that $\hat{s}t(a_1, a_2, \ldots, a_n, \lambda)\hat{s}^{-1} = t(a_{s(1)}, a_{s(2)}, \ldots, a_{s(n)}, \lambda)$. If $\mathbf{G} = GSpin_{2n+1}$, then denote by $\hat{c}_i$ a representative of the element of $W(\hat{G}, \hat{T})$ such that $\hat{c}_i t(a_1, \ldots, a_i, \ldots, a_n, \lambda)\hat{c}_i^{-1} = t(a_1, \ldots, \lambda a_i^{-1}, \ldots, a_n, \lambda)$. Then $W(\hat{G}, \hat{T})$ is generated by $\{\hat{s} | s \in S_n\}$ and $\{\hat{c}_i | 1 \leq i \leq n\}$. If $\mathbf{G} = GSpin_{2n}$, then $W(\hat{G}, \hat{T})$ is generated by $\{\hat{s} | s \in S_n\}$ and $\{c_i c_j | 1 \leq i, j \leq$

$n\}$. We have the pairing of roots $\Phi(\mathbf{G},\mathbf{T})$ and coroots $\Phi(\hat{G},\hat{T})$ which we denote by $\alpha \mapsto \check{\alpha}$, and denote by w the element of $W(\mathbf{G},\mathbf{T})$ corresponding to $\hat{w}$ by this pairing.

Let $\mathbf{P} = \mathbf{M}\mathbf{N} \supset \mathbf{B}$ be a standard parabolic subgroup of $\mathbf{G}$. Then, for some $\theta \subset \Delta$ we have $\mathbf{P} = \mathbf{P}_\theta = \mathbf{M}_\theta\mathbf{N}_\theta$. Then there is a partition $n = n_1 + n_2 + \cdots + n_r + m$, so that $\theta = \Delta \setminus \{\alpha_{n_1}, \alpha_{n_1+n_2,}, \dots, \alpha_{n_1+n_2+\cdots+n_r}, \alpha_n\}$, if $m = 0$, and $\theta = \Delta \setminus \{\alpha_{n_1}, \alpha_{n_1+n_2,}, \dots, \alpha_{n_1+n_2+\cdots+n_r}\}$, if $m > 0$. Then

$$\mathbf{M} \simeq GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r} \times \mathbf{G}_m. \tag{1.1}$$

Let $\mathbf{A}$ be the split component of $\mathbf{P}$, and let $\Phi(\mathbf{P},\mathbf{A})$ be the reduced roots of $\mathbf{A}$ in $\mathbf{P}$. For $i = 1, 2, \dots, r$, we let $k_i = n_1 + \cdots + n_i$. Then, for $1 \le i < j \le r$, set $\alpha_{ij} = e_{k_i} - e_{k_{j-1}+1}$, and $\beta_{ij} = e_{k_i} + e_{k_{j-1}+1}$, and

$$\gamma_i = \begin{cases} e_{k_i} + e_n & \text{if } \mathbf{G} = GSpin_{2n}; \\ e_{k_i} & \text{if } \mathbf{G} = GSpin_{2n+1}. \end{cases}$$

We describe the relative Weyl group $W_{\mathbf{M}} = N_{\mathbf{G}}(\mathbf{A}_M)/Z_{\mathbf{G}}(\mathbf{A}_M) = N_{\mathbf{G}}(\mathbf{A}_{\mathbf{M}})/\mathbf{M}$. Suppose $\mathbf{M}$ is as above. As in the case of other groups of classical type, $W_M \subset S_r \ltimes \mathbb{Z}_2^r$. If $\mathbf{G}$ is of type B_n, then $W_{\mathbf{M}} \simeq S \ltimes \mathbb{Z}_2^r$, for some subgroup S of S_r. In fact $S = \langle (ij) | i < j, n_i = n_j \rangle$. More precisely, let $k_0 = 0$, and for $i = 1, 2, \dots, r-1$, let k_i be as above. If $n_i = n_j$, let $[ij] \in W(\mathbf{G},\mathbf{T})$ be the element $\prod_{k=1}^{n_i} (k_{i-1} + k\, k_{j-1} + k)$. Then $[ij] \mapsto (ij)$ gives an isomorphism of $W_M \cap S_n$ to S. We generally denote these elements by the more standard (ij). For $1 \le i \le r$, we let $C_i = \prod_{k=1}^{n_i} c_{k_{i-1}+k}$. We call C_i a **block sign change**, and $\langle C_i | i = 1, \dots, r \rangle \simeq \mathbb{Z}_2^r$ is the sign change subgroup of $W_{\mathbf{M}}$. The action of S on $\mathbf{M}$ is given by

$$(ij) : (g_1, \dots, g_r, h) = (g_1, \dots, g_{i-1}, g_j, g_{i+1}, \dots, g_{j-1}, g_i, \dots, g_r, h).$$

Also, from the action of C_i on the root datum of $\mathbf{G}$ (see [4]) we have $C_i \cdot (g_1, \dots, g_i, \dots, g_r, h) = (g_1, \dots, {}^t g_i^{-1}, \dots, g_r, e_0^*(\det g_i)h)$. If $\mathbf{G}$ is of type D_n, then $W_{\mathbf{M}} \simeq S \ltimes \mathcal{C}$, where S is as above for type B_n, and $\mathcal{C} \subset \mathbb{Z}_2^r$. If $m = 0$, then we have $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, where $\mathcal{C}_1 = \langle C_i | n_i \text{ is even} \rangle$, and $\mathcal{C}_2 = \langle C_i C_j | n_i, n_j \text{ are odd} \rangle$. If $m > 0$, then $\mathcal{C} \simeq \mathbb{Z}_2^r$, and

$$\mathcal{C} = \langle C_i | n_i \text{ is even} \rangle \times \langle C_i c_n | n_i \text{ is odd} \rangle.$$

We note that S and each C_i acts as in the case of type B_n, (and of course $C_i C_j$ acts as the product in type D_n). In the case of $m > 0$ and n_i odd, we have $C_i c_n \cdot (g_1, \dots, g_i, \dots, g_r, h) =$

$(g_1, \ldots, {}^t g_i^{-1}, \ldots, g_r, (\det g_i)(c_n \cdot h))$, where c_n is given by the outer automorphism on the Dynkin diagram of $\mathbf{G}_m$.

2. R-groups for GSpin

We continue with the notation of the previous section. Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G} = \mathbf{G}_n$ and assume $\mathbf{M}$ is of the form (1.1). Let $\sigma \in \mathcal{E}_2(M)$. Then $\sigma \simeq \sigma_1 \otimes \sigma_2 \cdots \otimes \sigma_r \otimes \tau$, where $\sigma_i \in \mathcal{E}_2(GL_{n_i}(F))$, and $\tau \in \mathcal{E}_2(G_m)$. For $\alpha \in \Phi(\mathbf{P}, \mathbf{A})$, we set $\mathbf{A}_\alpha = (\mathbf{A} \cap \ker \alpha)^\circ$, and $\mathbf{M}_\alpha = Z_{\mathbf{G}}(\mathbf{A}_\alpha)$. Then ${}^*\mathbf{P}_\alpha = \mathbf{P} \cap \mathbf{M}_\alpha = \mathbf{M}\mathbf{N}_\alpha$, where $\mathbf{N}_\alpha = \mathbf{N} \cap \mathbf{M}_\alpha$ is a maximal parabolic subgroup of $\mathbf{M}_\alpha$ with Levi component $\mathbf{M}$. We let $W_\alpha = W(\mathbf{M}_\alpha, \mathbf{A})$. If $W_\alpha \neq \{1\}$, we let w_α be the unique nontrivial element of W_α. We recall the Plancherel measure, $\mu_\alpha(\sigma)$ is determined by the standard intertwining operator attached to $\mathrm{Ind}_{{}^*P_\alpha}^{M_\alpha}(\sigma)$, and in particular, $\mu_\alpha(\sigma) = 0$ if and only if $w_\alpha \sigma \simeq \sigma$ and $\mathrm{Ind}_{{}^*P_\alpha}^{M_\alpha}(\sigma)$ is irreducible.

We note if $\alpha = \alpha_{ij}$, then

$$M_\alpha \simeq \prod_{k \neq i,j} GL_{n_k} \times GL_{n_i+n_j} \times \mathbf{G}_m, \tag{2.1}$$

and

$$W_\alpha = \begin{cases} 1 & \text{if } n_i \neq n_j; \\ \{1, (ij)\} & \text{if } n_i = n_j. \end{cases}$$

If $\alpha = \beta_{ij}$, then $\mathbf{M}_\alpha \simeq \mathbf{M}_{\alpha_{ij}}$ is again given by (2.1), and

$$W_\alpha = \begin{cases} 1 & \text{if } n_i \neq n_j; \\ \{1, (ij)C_iC_j\} & \text{if } n_i = n_j. \end{cases}$$

Finally, for $\alpha = \gamma_i$, we have

$$\mathbf{M}_\alpha \simeq \prod_{k \neq i} GL_{n_k} \times \mathbf{G}_{n_i+m}.$$

If $\mathbf{G}$ is of type B_n, or n_i is even, then $W_\alpha = \{1, C_i\}$. If $\mathbf{G}$ is of type D_n, and n_i is odd, then

$$W_\alpha = \begin{cases} C_i c_n & \text{if } m > 0; \\ 1 & \text{if } m = 0. \end{cases}$$

We note, for $\mathbf{G}$ of type B_n,

$$w_\alpha \sigma \simeq \begin{cases} \sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes \sigma_j \otimes \sigma_{i+1} \cdots \otimes \sigma_{j-1} \otimes \sigma_i \otimes \cdots \otimes \sigma_r \otimes \tau; \\ \sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes (\tilde{\sigma}_j \otimes \omega_\tau) \otimes \sigma_{i+1} \cdots \sigma_{j-1} \otimes (\tilde{\sigma}_i \otimes \omega_\tau) \otimes \sigma_{j+1} \otimes \cdots \otimes \sigma_r \otimes \tau; \\ \sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes (\tilde{\sigma}_i \otimes \omega_\tau) \otimes \cdots \otimes \sigma_r \otimes \tau, \end{cases}$$

if $\sigma = \alpha_{ij}, \beta_{ij}$, or γ_i, respectively. Here ω_τ is the central character of τ restricted to the identity component of the center of G_m. For type D_n, the result is as above, except in the case where $\alpha = \gamma_i$, n_i is odd and $m > 0$, in which case

$$w_\alpha \sigma \simeq \sigma_1 \otimes \cdots \otimes (\tilde{\sigma}_i \otimes \omega_\tau) \otimes \cdots \otimes \sigma_r \otimes (c_n \cdot \tau).$$

Lemma 2.1. *For $1 \le i < j \le r-1$ we have $\operatorname{Ind}_{*P_{\alpha_{ij}}}^{M_{\alpha_{ij}}}(\sigma)$ is irreducible. Similarly $\operatorname{Ind}_{*P_{\beta_{ij}}}^{M_{\beta_{ij}}}(\sigma)$ is irreducible.*

Proof. Let $\alpha = \alpha_{ij}$. In this case M_α is given by (2.1). Let $\mathbf{Q}_{ij}$ be the standard $GL_{n_i} \times GL_{n_j}$–parabolic subgroup of $GL_{n_i+n_j}$. Then,

$$\operatorname{Ind}_{*P_\alpha}^{M_\alpha}(\sigma) \simeq \left(\bigotimes_{\ell \neq i, j+1} \sigma_\ell \right) \otimes \left(\operatorname{Ind}_{Q_{ij}}^{GL_{n_i+n_j}(F)}(\sigma_i \otimes \sigma_j) \right) \otimes \tau,$$

and the result now follows from Olsanskii or Bernstein and Zelevinski[8, 24].

If $\alpha = \beta_{ij}$, then we again have M_α is given by (2.1), and in this case

$$\operatorname{Ind}_{*P_\alpha}^{M_\alpha}(\sigma) \simeq \left(\bigotimes_{\ell \neq i, j} \sigma_\ell \right) \otimes \left(\operatorname{Ind}_{Q_{ij}}^{GL_{n_i+n_j}(F)}(\sigma_i \otimes (\tilde{\sigma}_j \otimes \omega_\tau)) \right) \otimes \tau.$$

Thus, the result again follows from [8, 24]. □

From this we derive the following result.

Corollary 2.2. *If $\alpha = \alpha_{ij}$, then $\mu_\alpha(\sigma) = 0$ if and only if $n_i = n_j$ and $\sigma_i \simeq \sigma_j$. If $\alpha = \beta_{ij}$, then $\mu_\alpha(\sigma) = 0$ if and only if $n_i = n_j$ and $\sigma_i \simeq \tilde{\sigma}_j \otimes \omega_\tau$.*

Lemma 2.3. *Let $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r \otimes \tau \in \mathcal{E}_2(M)$, and let $R = R(\sigma)$. Suppose $w \in R$ and $w = sc$, with $s \in S_r$ and $c \in \mathbb{Z}_2^r$. Then $s = 1$.*

Proof. This is a **Keys argument** as defined in [12] and introduced in [19]. Since the sign changes act independently on the disjoint cycles of s, we may suppose, without loss of generality, that $s = (12 \cdots j)$. Furthermore, if $\mathbf{G}$ is of type B_n, then up to conjugation by sign changes we may

assume $c = C_j c'$, or $c = c'$,with c' not changing signs among $1, 2, \dots, j$. If $\mathbf{G}$ is of type D_n, then we may assume c is either of the same form, or of the form $C_{j-1}C_j c'$, with c' changing no signs among $1, 2, \dots, j$. If c changes no (block) signs among $1, 2, \dots, j$, then we note that $\sigma_1 \simeq \sigma_2 \simeq \cdots \simeq \sigma_j$. So, in particular $\alpha_{1j} \in \Delta'$ and $w(\alpha_{1j}) = -\alpha_{12} < 0$, so $w \notin R(\sigma)$. If $c = C_j c'$, then $\sigma_1 \simeq \sigma_2 \simeq \cdots \simeq \sigma_{j-1} \simeq \sigma_j \simeq (\tilde{\sigma}_1 \otimes \omega_\tau)$ and thus $\beta_{1j} \in \Delta'$. However, $w\beta_{1j} = -\alpha_{12} < 0$, so $w \notin R$. Finally, if $c = C_i C_j c'$, then $w\sigma \simeq \sigma$ implies $\sigma_1 \simeq \sigma_2 \simeq \cdots \simeq \sigma_{j-1} \simeq (\tilde{\sigma}_j \otimes \omega_\tau)$, and therefore, again, $\beta_{1j} \in \Delta'$, with $w\beta_{1j} = -\alpha_{12} < 0$, showing $w \notin R$. □

Corollary 2.4. *For $G = GSpin_{2n}$ or $GSpin_{2n+1}$, we have $R \subset \mathbb{Z}_2^r$.*

We let $W(\sigma) = \{w \in W(\mathbf{G}, \mathbf{A_M}) | w\sigma \simeq \sigma\}$. If $\mathbf{G}$ is of type B_n, and $W(\sigma) \neq 1$, then one of the following holds:

$$\sigma_i \simeq \sigma_j, \text{ for some } i \neq j; \tag{2.2}$$

$$\sigma_i \simeq \tilde{\sigma}_j \otimes \omega_\tau \text{ for some } i \neq j; \textit{and} \tag{2.3}$$

$$\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau. \tag{2.4}$$

Note that (2.2) holds if $(ij) \in W(\sigma)$, (2.3) holds if $(ij)C_iC_j \in W(\sigma)$, while (2.4) holds if $C_i \in W(\sigma)$. Also notice if $w = (ij)C_i \in W(\sigma)$, then $w^2 = C_iC_j \in W(\sigma)$, so this case is covered by (2.4). For $w \in W(\mathbf{G}, \mathbf{A})$, we let $R(w) = \{\alpha \in \Phi(\mathbf{P}, \mathbf{A}) | w\alpha < 0\}$.

For $B \subset \{1, 2, \dots, r\}$, we let $C_B = \prod_{i \in B} C_i$. If $C_B \in R(\sigma)$, then $R(C_B) \cap \Delta' = \emptyset$. Note that

$$R(C_B) = \left\{\alpha_{ij}, \beta_{ij} \middle| i \in B, i < j\right\} \cup \{\gamma_i | i \in B\}.$$

We let $\mathbf{Q}_i$ be the standard $GL_{n_i} \times \mathbf{G}_m$ parabolic subgroup of $\mathbf{G}_{n_i+m}$.

Theorem 2.5. *Let $\mathbf{G} = GSpin_{2n+1}$ and $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times \mathbf{G}_m$, with $m + \sum_i n_i = n$. Let $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \tau \in \mathcal{E}_2(M)$, with each $\sigma_i \in \mathcal{E}_2(GL_{n_i}(F))$ and $\tau \in \mathcal{E}_2(G_m)$. Let d be the number of nonequivalent classes among $\{\sigma_1, \dots, \sigma_r\}$ for which $\mathrm{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau)$ is reducible. Then $R(\sigma) \simeq \mathbb{Z}_2^d$. More precisely, let*

$$\Omega(\sigma) = \left\{i \middle| \mathrm{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau) \textit{ is reducible, and } \sigma_j \not\simeq \sigma_i \textit{ for all } j > i\right\}.$$

Then $R(\sigma) = \langle C_i \rangle_{i \in \Omega(\sigma)}$.

Remark 2.6. *By Bruhat Theory we know if* $\mathrm{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau)$ *is reducible implies* $C_i \in W(\sigma)$*, so* $\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau$.

Proof. From Corollary 2.4 we know $R \subset \langle C_i \rangle_{i=1}^r \simeq \mathbb{Z}_2^r$. Suppose $B \subset \{1, 2, \ldots, r\}$, with $C_B \in R(\sigma)$. Then $C_B \in W(\sigma)$, so $\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau$, for all $i \in B$. Thus, for each $i \in B$, we have $C_i \in W(\sigma)$. Since $R(C_i) \subset R(C_B)$, and $R(C_B) \cap \Delta' = \emptyset$, we have $R(C_i) \cap \Delta' = \emptyset$. So $C_i \in R(\sigma)$. Therefore, for some subset, Ω, of $\{1, 2, \ldots, r\}$ we have $R(\sigma) = \langle C_i | i \in \Omega \rangle$. Now suppose $C_i \in R(\sigma)$. For each $j > i$, we have $\alpha_{ij} \in R(C_i)$, and thus $\alpha_{ij} \notin \Delta'$. By Corollary 2.2 this implies $\sigma_j \not\simeq \sigma_i$, for all $j > i$. Also, for each $j > i$, we have $\beta_{ij} \in R(C_i)$, so $\sigma_j \not\simeq \tilde{\sigma}_i \otimes \omega_\tau$. However, since $\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau$, we see $\beta_{ij} \notin \Delta'$ imposes no further condition. Finally, since $\gamma_i \in R(C_i)$, we must have $\gamma_i \notin \Delta'$. We note

$$\mathbf{M}_{\gamma_i} \simeq \prod_{j \neq i} GL_{n_j} \times \mathbf{G}_{n_i+m},$$

and

$$\mathrm{Ind}_{{}^*P_{\gamma_i}}^{M_{\gamma_i}} \sigma \simeq \bigotimes_{j \neq i} \sigma_j \otimes \mathrm{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau).$$

Since $C_i \in W(\sigma) \cap W_{\gamma_i}$, we have $\gamma_i \notin \Delta'$ if and only if $\mathrm{Ind}_{Q_i}^{G_{n_i+m}}(\sigma \otimes \tau)$ is reducible. Thus, $i \in \Omega(\sigma)$, so $\Omega \subset \Omega(\sigma)$. Conversely, if $i \in \Omega(\sigma)$, then $C_i \sigma \simeq \sigma$, and $R(C_i) \cap \Delta' = \emptyset$, so $C_i \in \Omega$. Thus $\Omega = \Omega(\sigma)$, and $R(\sigma)$ has the form we claim. $\square$

Now suppose $\mathbf{G}$ is of type D_n. Let $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times \mathbf{G}_m$. We may assume n_i is even for $i = 1, 2, \ldots, t$, and n_i is odd for $i = t+1, \ldots, r$. If $m = 0$, then

$$\mathcal{C} \simeq \begin{cases} \mathbb{Z}_2^{r-1} & \text{if } t < r; \\ \mathbb{Z}_2^r & \text{otherwise.} \end{cases}$$

If $m > 0$, then $\mathcal{C} \simeq \mathbb{Z}_2^r$, as described above. If $m = 0$ or $c_n \tau \not\simeq \tau$, then the following describes the conditions under which $W(\sigma) \neq \{1\}$:

$$\sigma_i \simeq \sigma_j \text{ for some } i \neq j; \tag{2.5}$$

$$\sigma_i \simeq \tilde{\sigma}_j \otimes \omega_\tau \text{ for some } i \neq j; \tag{2.6}$$

$$\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau \text{ for some } i \text{ with } n_i \text{ even}; \tag{2.7}$$

$$\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau \text{ and } \sigma_j \simeq \tilde{\sigma}_j \otimes \omega_\tau \text{ for some } i \neq j \text{ with } n_i, n_j \text{ odd.} \tag{2.8}$$

We have (2.5) holding if and only if $(ij) \in W(\sigma)$, (2.6) holds if and only if $(ij)C_iC_j \in W(\sigma)$, while (2.7) and (2.8) are the conditions for either C_i (for n_i even) or C_iC_j to be in $W(\sigma)$. If $m > 0$ and

$c_n\tau \simeq \tau$, then (2.5),(2.6), and (2.7) are the conditions, with the restriction on parity removed from (2.7).

Theorem 2.7. *Let* $\mathbf{G} = Gspin_{2n}$*, and* $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times \mathbf{G}_m$*, with* $m + \sum_i n_i = n$*. Let* $\sigma \in \mathcal{E}_2(M)$*, with each* $\sigma_i \in \mathcal{E}_2(GL_{n_i}(F))$*, and* $\tau \in \mathcal{E}_2(G_m)$*.*

(i) If $m = 0$ *or* $c_n\tau \not\simeq \tau$*, then we let*

$$\Omega_1(\sigma) = \{1 \le i \le r | n_i \textit{ is even, } \operatorname{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau) \textit{ is reducible, and } \sigma_j \not\simeq \sigma_i \textit{ for all } i > j\},$$

and

$$\Omega_2(\sigma) = \{1 \le i \le r | n_i \textit{ is odd, } \sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau, \textit{ and } \sigma_j \not\simeq \sigma_i, \textit{ for all } j > i\}.$$

We set $d_i = |\Omega_i(\sigma)|$*, for* $i = 1, 2$*. Then* $R(\sigma) \simeq \mathbb{Z}_2^{d_1+d_2-1}$*, unless* $d_2 = 0$*, in which case* $R(\sigma) \simeq \mathbb{Z}_2^{d_1}$*. More precisely,*

$$R(\sigma) = \left\langle C_i \middle| i \in \Omega_1(\sigma) \right\rangle \times \left\langle C_iC_j \middle| i, j \in \Omega_2(\sigma) \right\rangle.$$

(ii) If $m > 0$ *and* $c_n\tau \simeq \tau$*, we let*

$$\Omega(\sigma) = \{1 \le i \le r | \operatorname{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau) \textit{is reducible, and } \sigma_j \not\simeq \sigma_i \textit{ for all } j > i\}.$$

Let $d = |\Omega(\sigma)|$*. Then* $R(\sigma) \simeq \mathbb{Z}_2^d$*, and in particular,*

$$R(\sigma) = \langle C_i | i \in \Omega(\sigma) \textit{ and } n_i \textit{ is even}\rangle \times \langle C_ic_n | i \in \Omega(\sigma) \textit{ and } n_i \textit{ is odd}\rangle.$$

Proof. We assume n_i is even for $i = 1, 2, \ldots, t$, and n_i is odd for $i = t+1, \ldots, r$. Suppose $m = 0$. Then $W_{\mathbf{M}} = S \ltimes \mathcal{C}$, where

$$\mathcal{C} = \langle C_i | 1 \le i \le t\rangle \times \langle C_iC_j | t+1 \le i, j \le r\rangle.$$

By Corollary 2.4, $R(\sigma) \subset \mathcal{C}$. Suppose $B \subset \{1, 2, \ldots r\}$. Then we let $B_1 = B \cap \{1, 2, \ldots, t\}$, and $B_2 = B \setminus B_1$. Suppose $C_B = \prod_{i \in B} C_i \in R(\sigma)$. Then $\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau$, for each $i \in B$. Thus, $C_i \in W(\sigma)$, for each $i \in B_1$, and $C_iC_j \in W(\sigma)$, for each $i, j \in B_2$. As in the case of type B_n, we have, for each $i \in B$, $R(C_i) \subset R(C_B)$, and thus $C_i \in R(\sigma)$, for each $i \in B_1$, and $C_iC_j \in R(\sigma)$ for each $i, j \in B_2$. Thus, there is some $\Omega \subset \{1, \ldots, r\}$, for which

$$R(\sigma) = \langle C_i | \, i \in \Omega_1\rangle \times \langle C_iC_j | \, i, j \in \Omega_2\rangle.$$

For $1 \le i \le t$, we have

$$R(C_i) = \{\gamma_i\} \cup \{\alpha_{ij}, \beta_{ij}\}_{j>i}.$$

We have $R(C_i) \cap \Delta' = \emptyset$, so by Corollary 2.2 $\sigma_j \not\simeq \sigma_i$ for all $j > i$, as in the case of type B_n. Further note, since $C_i \in W(\sigma)$, we have $\gamma_i \in \Delta'$ if and only if $\mathrm{Ind}_{{}^*P_{\gamma_i}}^{M_{\gamma_i}} \sigma$ is irreducible. Since

$$\mathrm{Ind}_{{}^*P_{\gamma_i}}^{M_{\gamma_i}} \sigma \simeq \left(\bigotimes_{j \neq i} \sigma_i \right) \otimes \mathrm{Ind}_{Q_i}^{G_{n_i+m}} (\sigma_i \otimes \tau),$$

we see $C_i \in R(\sigma)$ implies $\mathrm{Ind}_{Q_i}^{G_{n_i+m}} (\sigma_i \otimes \tau)$ is reducible. Thus, $i \in \Omega_1(\sigma)$. Therefore, we have $\Omega_1 \subset \Omega_1(\sigma)$. However, the reverse inclusion is now obvious.

Now suppose $i, j \geq t+1$, and $C_i C_j \in R(\sigma)$. Then we have noted $\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau$, and $\sigma_j \simeq \tilde{\sigma}_j \otimes \omega_\tau$. Note further,

$$R(C_i C_j) = \{\gamma_i, \gamma_j\} \cup \{\alpha_{ik}, \beta_{ik}\}_{k>i} \cup \{\alpha_{j\ell}, \beta_{j\ell}\}_{\ell > j}.$$

As above, this now implies $\sigma_i \not\simeq \sigma_k$, for $k > i$, and $\sigma_j \not\simeq \sigma_\ell$, for $\ell > j$. Thus, we see $i, j \in \Omega_2(\sigma)$, so $\Omega_2 \subset \Omega_2(\sigma)$. For the opposite inclusion we note, $W_{\mathbf{M}_{\gamma_i}} = \{1\} = W_{\mathbf{M}_{\gamma_j}}$, and hence $\gamma_i, \gamma_j \notin \Delta'$. Thus, if $i, j \in \Omega_2(\sigma)$, then $C_i C_j \in R(\sigma)$. Therefore, $R(\sigma)$ has the form we claim.

If $m > 0$ and $c_n \tau \not\simeq \tau$, then the argument above is essentially valid with the following adjustments. We note $W_{\mathbf{M}} = S \ltimes \mathcal{C}$, with

$$\mathcal{C} = \langle C_i | 1 \leq i \leq t \rangle \times \langle C_i c_n | i > t \rangle, \tag{2.9}$$

and since $c_n \tau \not\simeq \tau$, we have $C_i c_n \notin W(\sigma)$, for $i > t$. Also, we note for $i > t$, $W_{\mathbf{M}_{\gamma_i}} = \{1, C_i c_n\}$, so $W_{\mathbf{M}_{\gamma_i}} \cap W(\sigma) = \{1\}$, and again we must have $\gamma_i \notin \Delta'$.

(ii) Now suppose $m > 0$ and $c_n \tau \simeq \tau$. We still have $W_{\mathbf{M}} = S \ltimes \mathcal{C}$, with $\mathcal{C}$ given by (2.9). For $i = 1, 2, \ldots, r$, we let

$$\bar{C}_i = \begin{cases} C_i & \text{if } i \leq t; \\ C_i c_n & \text{if } i > t. \end{cases}.$$

If $B \subset \{1, 2, \ldots, r\}$, and $\bar{C}_B = \prod_{i \in B} \bar{C}_i \in R(\sigma)$, then $\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau$, for each $i \in B$. So $\bar{C}_i \in W(\sigma)$, for each $i \in B$. Further,

$$R(\bar{C}_B) = \bigcup_{i \in B} R(\bar{C}_i),$$

so $\bar{C}_i \in R(\sigma)$ for each $i \in B$. Thus, there is some $\Omega \subset \{1, 2, \ldots, r\}$ such that $R(\sigma) = \left\langle \bar{C}_i \mid i \in \Omega \right\rangle$. Since

$$R(\bar{C}_i) = \{\alpha_{ij}, \beta_{ij}\}_{j > i} \cup \{\gamma_i\},$$

and, given $\bar{C}_i \in W(\sigma)$, we have $\alpha_{ij}, \beta_{ij} \in \Delta'$ if and only if $\sigma_i \simeq \sigma_j$. Further, as above, $\gamma_i \in \Delta'$ if and only if $\bar{C}_i \in W(\sigma)$, and $\mathrm{Ind}_{Q_i}^{G_{n_i+m}}(\sigma \otimes \tau)$ is irreducible. Thus,

$$\Omega = \{i | \mathrm{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau) \text{ is reducible, and } \sigma_j \not\simeq \sigma_i, \text{ for all } j > i\} = \Omega(\sigma),$$

as claimed. □

3. Elliptic Representations for *GSpin* groups

We now consider the question of which tempered representations of $G = GSpin_n(F)$ are elliptic. We can adapt the arguments of [18] to our current situation. We let G_e be the set of regular elliptic elements of G. If π is an irreducible representation of G, then we denote by Θ_π its character. By Harish-Chandra [15] we know Θ_π is given by a locally integrable function, also denoted Θ_π, on the regular set. We let Θ_π^e be the restriction of Θ_π to G_e. Then $\pi \in \mathcal{E}_t(G)$ is elliptic if $\Theta_\pi^e \neq 0$.

We begin by showing the 2–cocyle arising from constructing self intertwining operators in $\mathcal{C}(\sigma)$ is a coboundary. Let $\mathbf{G}_n = GSpin_{2n}$ or $GSpin_{2n+1}$. Suppose $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times \mathbf{G}_m$ is a proper Levi subgroup of $\mathbf{G}$. Let $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r \otimes \tau$ be an irreducible discrete series of M. Let V be the space of the representation σ. For each $w \in R(\sigma)$, we choose an intertwining operator $T_w : V \to V$ so that $T_w \circ w\sigma = \sigma \circ T_w$.

Lemma 3.1. *We can choose the operators T_w so that $T_{w_1 w_2} = T_{w_1} T_{w_2}$.*

Proof. For each i, we let V_i be the space of the representation σ_i. So $V = V_1 \otimes \cdots V_r \otimes V_\tau$. Denote by σ_i^* the representation on V_i given by $\sigma_i^*(g) = \sigma_i({}^t g^{-1})$. By the work of Gelfand and Kazhdan [11] we know $\sigma_i^* \simeq \tilde{\sigma}_i$. Let $\mathcal{B}(\sigma) = \{i | \sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau\}$. For each $i \in \mathcal{B}(\sigma)$, we choose an intertwining operator $T_i : V_i \to V_i$, with $T_i(\sigma_i^* \otimes \omega_\tau) = \sigma_i T_i$. We note T_i^2 is a scalar on V_i, and so we can choose T_i so that $T_i^2 = 1$. Extend this to an operator on V, by setting T_i^V to be trivial on each factor, except for V_i, where it is T_i. Now $T_i^V \circ C_i \sigma = \sigma T_i^V$, and $(T_i^V)^2 = \mathrm{Id}$. Also note, for $i \neq j$, we have $T_i^V T_j^V = T_j^V T_i^V$. If $\mathbf{G}$ is of type D_n, and $c_n \tau \simeq \tau$, we choose T_τ intertwining τ and $c_n\tau$, again with $T_\tau^2 = \mathrm{Id}$. Extend T_τ to V by setting T_τ^V to be trivial on each V_i and to be T_τ on V_τ. Suppose $B \subset \mathcal{B}(\sigma)$, and that

$$w = C_B = \prod_{i \in B} C_i \in R(\sigma).$$

Then, we set

$$T_w = \prod_{i \in B} T_i^V.$$

In the case where $\mathbf{G}$ is of type D_n and $c_n\tau \simeq \tau$, we may have

$$w = \bar{C}_B = \left(\prod_B C_i\right) c_n \in R(\sigma),$$

in which case we set

$$T_w = \left(\prod_{i\in B} T_i^V\right) T_\tau^V.$$

We then see that for $C_B, C_{B'} \in R(\sigma)$, we have

$$T_{C_B}T_{C_{B'}} = \prod_B T_i^V \prod_{B'} T_j^V = \prod_{B\wedge B'} T_i^V,$$

where $B \wedge B'$ is the symmetric difference. Since $C_B C_{B'} = C_{B\wedge B'}$, we have the result in this case. A similar argument shows, in the case where $\mathbf{G} = D_n$ and $c_n\tau \simeq \tau$, that

$$T_{\bar{C}_B}T_{C_{B'}} = T_{\bar{C}_{B\wedge B'}} = T_{\bar{C}_B C_{B'}},$$

and

$$T_{\bar{C}_B}T_{\bar{C}_{B]}} = T_{C_{B\wedge B'}} = T_{\bar{C}_B\bar{C}_{B'}}.$$

Thus, we have the claim. $\square$

Since the cocycle $\eta : R(\sigma) \times R(\sigma) \to \mathbb{C}$ is determined by $T_{w_1w_2} = \eta(w_1, w_2)T_{w_1}T_{w_2}$ we have η is a coboundary, and immediately get the following result.

Corollary 3.2. *For any Levi subgroup* $\mathbf{M}$ *of* $\mathbf{G}_n$ *and any* $\sigma \in \mathcal{E}_2(M)$, *we have* $\mathcal{C}(\sigma) \simeq \mathbb{C}[R(\sigma)]$, *so* $i_{G,M}(\sigma)$ *decomposes with multiplicity one.*

Now, let $\mathbf{A} = \mathbf{A}_\theta$ be the split component of $\mathbf{M}$, and let $\mathfrak{a} = \mathfrak{a}_\theta$ be its real Lie algebra. If $\sigma \in \mathcal{E}_2(M)$, and $w \in R(\sigma)$, then we let $\mathfrak{a}_w = \{H \in \mathfrak{a} | w \cdot H = H\}$. We let Z be the split component of G and $\mathfrak{z}$ be its real Lie algebra. Now, by Theorem 1.1 of [18], we know $i_{G,M}(\sigma)$ has elliptic components if and only if there is a $w \in R(\sigma)$ with $\mathfrak{a}_w = \mathfrak{z}$. Further, if $\mathfrak{a}_{R(\sigma)} = \cap_{w\in R(\sigma)}\mathfrak{a}_w$, then each component of $i_{G,M}(\sigma)$ is irreducibly induced from an elliptic tempered representation if there is some $w \in R(\sigma)$ so that $\mathfrak{a}_R = \mathfrak{a}_w$.

Theorem 3.3. *Let* $\mathbf{G} = GSpin_{2n+1}$, *and suppose* $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times \mathbf{G}_m$, *and* $\sigma \in \mathcal{E}_2(M)$. *Then* $\mathrm{Ind}_P^G(\sigma)$ *has elliptic constituents if and only if* $R(\sigma) \simeq \mathbb{Z}_2^r$. *Any* $\pi \in \mathcal{E}_t(G)$, *is either elliptic, or there is a choice of* $\mathbf{M}'$ *and an irreducible elliptic tempered representation* σ *of* M' *with* $\pi = \mathrm{Ind}_{P'}^G(\sigma)$.

Proof. We will use the explicit realization of $R(\sigma)$ we developed in Theorem 2.5. Suppose $R \simeq \mathbb{Z}_2^d$. Let $\mathfrak{a} = \mathfrak{a}_M$. Then we can identify $\mathfrak{a}$ with $\{(x_1, x_2, \ldots, x_r, y)|x_i, y) \in \mathbb{R}\}$, and note, under this identificaton $\mathfrak{z} = \{(y/2, \ldots, y/2, y)|y \in \mathbb{R}\}$. $\mathcal{C}$ acts on $\mathfrak{a}$ by

$$C_i(x_1, \ldots x_{i-1}, x_i, x_{i+1}, \ldots, x_r, y) = (x_1, \ldots, x_{i-1}, y - x_i, x_{i+1}, \ldots, x_r, y).$$

Thus, if $C = C_B$, as above, then $\mathfrak{a}_C = \{(x_1, \ldots, x_r, y)|x_i = y/2, \forall i \in B\}$ Without loss of generality, we may assume $R(\sigma) = \langle C_r, C_{r-1}, \ldots, C_{r-d+1}\rangle$. Let $w_0 = C_{r-d+1}C_{r-d+2}\cdots C_r$. Note, for each $w \in R(\sigma)$, we have $\mathfrak{a}_{w_0} \subset \mathfrak{a}_w$, and thus $\mathfrak{a}_{R(\sigma)} = \mathfrak{a}_{w_0}$. Now, $\mathfrak{a}_{w_0} = \mathfrak{z}$ if and only if $w_0 = C_1C_2\cdots C_r$, and thus, by [2, 18] $\mathrm{Ind}_P^G(\sigma)$ has elliptic constituents if and only if $R(\sigma) \simeq \mathbb{Z}_2^r$. In this case, every component of $\mathrm{Ind}_P^G(\sigma)$ is elliptic. The last statement of the claim follows from the fact $\mathfrak{a}_{R(\sigma)} = \mathfrak{a}_{w_0}$, and Lemma 1.3 of [18]. □

Theorem 3.4. *Let $\mathbf{G} = GSpin_{2n}$, and $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times \mathbf{G}_m$. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \tau \in \mathcal{E}_2(M)$.*

(i) Suppose $m = 0$ or $c_n\tau \not\simeq \tau$. We let $\Omega_1(\sigma), \Omega_2(\sigma), d_1, d_2$, and d be defined as in Theorem 2.7. Then $\mathrm{Ind}_P^G(\sigma)$ has elliptic components if and only if $d = r$ and d_2 is even, in which case every component is elliptic. If $\pi \subset \mathrm{Ind}_P^G(\sigma)$ is not elliptic, then $\pi \simeq \mathrm{Ind}_{P'}^G(\sigma')$ for some elliptic representation σ' of a Levi subgroup M' of G if and only if d_2 is even or $d_2 = 1$.

(ii) Suppose $m > 0$ and $c_n\tau \simeq \tau$. Let $R(\sigma) \simeq \mathbb{Z}_2^d$. Then $\mathrm{Ind}_P^G(\sigma)$ has elliptic components if and only if $d = r$, in which case all components are elliptic. Furthermore, for any $\pi \in \mathcal{E}_t(G)$ there is a Levi subgroup $\mathbf{M}'$ of $\mathbf{G}$, and an irreducible elliptic tempered representation σ' of M' so that $\pi \simeq \mathrm{Ind}_{P'}^G(\sigma')$.

Proof. (i) As in Theorem 2.7 we assume $\Omega_1(\sigma) = \{r - d_1 + 1, r - d_1 + 2, \ldots, r\}$, and $\Omega_2(\sigma) = \{r - d + 1, r - d + 2, \ldots, r - d_1\}$. Then

$$R(\sigma) = \langle C_iC_j|i, j \in \Omega_2(\sigma)\rangle \times \langle C_i|i \in \Omega_1(\sigma)\rangle.$$

We note $\mathfrak{a} = \mathfrak{a}_M$ can be identified with $\{(x_1, \ldots, x_r, y)\,|x_i, y \in \mathbb{R}\}$, in such a way so $C_i \cdot (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_r, y) = (x_1, \ldots, x_{i-1}, y - x_i, x_{i+1}, \ldots, x_r, y)$. So, if $d_2 \neq 1$, we have

$$\mathfrak{a}_{R(\sigma)} = \left\{(x_1, \ldots, x_r, y)|x_i = \frac{y}{2} \text{ for all } r - d + 1 \leq i \leq r\right\},$$

while if $d_2 = 1$, then

$$\mathfrak{a}_{R(\sigma)} = \left\{(x_1, \ldots, x_r, y)|x_i = \frac{y}{2} \text{ for all } r - d_1 + 1 \leq i \leq r\right\}.$$

If d_2 is even then $w_0 = C_{r-d+1}C_{r-d+2}\cdots C_r \in R(\sigma)$, and $\mathfrak{a}_{w_0} = \mathfrak{a}_{R(\sigma)}$. If $d_2 = 1$, then, $w_0 = C_{r-d_1+1}C_{r-d+2}\cdots C_r \in R(\sigma)$, and again $\mathfrak{a}_{w_0} = \mathfrak{a}_{R(\sigma)}$. Thus, in either of these cases, we have each component must be irreducibly induced from an elliptic tempered representation of some Levi subgroup [18]. On the other hand, if $d_2 \geq 3$ and d_2 is odd, then, for any $w \in R(\sigma)$ we have $\mathfrak{a}_{R(\sigma)} \subsetneq \mathfrak{a}_w$, so components of these induced representations are not irreducibly induced from elliptic representations. Finally, since $\mathfrak{z}$ is identified with $\{(y/2, y/2, \ldots, y/2, y)|y \in \mathbb{R}\}$, then we see $\mathrm{Ind}_P^G(\sigma)$ has elliptic components if and only if $C_1C_2\ldots C_r \in R(\sigma)$, which occurs if and only if $d = r$ and d_2 is even.

(ii) Now suppose $m > 0$ and $c_n\tau \simeq \tau$. We let $\Omega(\sigma)$ be defined as in Theorem 2.7. We assume, without loss of generality, $\Omega(\sigma) = \{r-d+1, \ldots, r\}$. Then

$$R(\sigma) = \langle C_i|r-d+1 \leq i \leq r \text{ and } n_i \text{ is even}\rangle \times \langle C_ic_n|r-d+1 \leq i \leq r \text{ and } n_i \text{ is odd}\rangle.$$

Let $d_2 = \{i|r-d+1 \leq i \leq r \text{ and } n_i \text{ is odd}\}$, and

$$w_0 = \begin{cases} C_{r-d+1}C_{r-d+2}\cdots C_r & \text{if } n_i \text{ is even;} \\ C_{r-d+1}C_{r-d+2}\cdots C_rc_n & \text{if } n_i \text{ is odd.} \end{cases}$$

With the identification of $\mathfrak{a} = \mathfrak{a}_M$ with $\mathbb{R}^{r+1}$ as above, we have

$$\mathfrak{a}_{w_0} = \{(x_1 \ldots, x_r, y)\,|x_i = y/2 \text{ for all } r-d+1 \leq i \leq r\}\,.$$

Note, for any $w \in R(\sigma)$ we have $\mathfrak{a}_{w_0} \subset \mathfrak{a}_w$, so $\mathfrak{a}_{R(\sigma)} = \mathfrak{a}_{w_0}$. Now, $\mathfrak{a}_{w_0} = \mathfrak{z}$ if and only if $d = r$. Thus, the elliptic spectrum is as claimed, and the tempered spectrum is irreducibly induced from the elliptic spectra of the Levi subgroups.

□

Now we assume $\mathbf{G} = \mathbf{G}_n = GSpin_{2n}$ or $GSpin_{2n+1}$. Denote $R = R(\sigma)$, and let $\hat{R}$ be the set of irreducible characters of R. We let $\kappa \leftrightarrow \pi_\kappa$ be the correspondence between $\hat{R}$ and the (equivalence classes of) irreducible components of $\mathrm{Ind}_P^G(\sigma)$ described by Keys [20] (see also Arthur [2] and Herb [18]). Suppose $\mathrm{Ind}_P^G(\sigma)$ has elliptic components, as described in Theorems 3.3 and 3.4. Then either $C_1C_2\ldots C_r \in R$ or $C_1C_2\ldots C_rc_n \in R$. Let

$$C_0 = \begin{cases} C_1C_2\ldots C_rc_n, & \text{if } \mathbf{G} = GSpin_{2n}, d_2 \text{ is odd , and } c_n\tau \simeq \tau; \\ C_1C_2\ldots C_n, & \text{otherwise.} \end{cases}$$

For $\kappa \in \hat{R}$ we let $\varepsilon(\kappa) = \kappa(C_0)$.

Theorem 3.5. *Suppose* $\mathbf{G} = GSpin_{2n}$ *or* $GSpin_{2n+1}$. *Let* $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times \mathbf{G}_m$ *be a Levi subgroup and suppose* $\sigma = \sigma_1 \otimes \ldots \sigma_r \otimes \tau \in \mathcal{E}_2(M)$. *Suppose* $\mathrm{Ind}_P^G(\sigma)$ *has elliptic components. Let* $\kappa \in \hat{R}$. *Then* $\Theta^e_{\pi_\kappa} = \kappa(C_0)\Theta^e_{\pi_1}$.

Proof. First suppose $\mathbf{G} = GSpin_{2n+1}$, or $c_n\tau \simeq \tau$. For $1 \leq i \leq r$, we let $\mathbf{M}_i$ be the Levi subgroup of $\mathbf{G}$ of the form $GL_{n_i} \times \mathbf{G}_{n-n_i}$. Let $\mathbf{N}_i = \mathbf{M}_i \cap \mathbf{N}$, and $\mathbf{P}_i = \mathbf{M}\mathbf{N}_i$. We let $R_i = R_i(\sigma)$ be the R–group attached to $\mathrm{Ind}_{P_i}^{M_i}(\sigma)$. Since we are assuming $\mathrm{Ind}_P^G(\sigma)$ has elliptic components, we know $\Delta' = \emptyset$. Thus, the compatibility condition in Section 2 of [2] is satisfied (see also [18]). Thus, we can identify R_i with the subgroup of R generated by $\{C_j | 1 \leq j \leq r, j \neq i\}$, or $\{\bar{C}_j | 1 \leq j \leq r, j \neq i\}$, where $\bar{C}_i$ is define as in the proof of Theorem 2.7. We now combine these situations by letting $R = \langle D_i | 1 \leq i \leq r\rangle$, where $D_i = C_i$ or $\bar{C}_i$, in the obvious way. Let $\eta \leftrightarrow \rho_\eta$ be the correspondence between $\hat{R}_i$ and components of $\mathrm{Ind}_{P_i}^{M_i}(\sigma)$. If $\eta \in \hat{R}_i$, we let $\hat{R}(\eta) = \{\kappa \in \hat{R} | \kappa|_{R_i} = \eta\}$. Then $\hat{R}(\eta) = \{\eta^+, \eta^-\}$, where $\eta^\pm(D_j) = \eta(D_j)$, for $i \neq j$, and $\eta^\pm(D_i) = \pm 1$. By Arthur [2] we have $\mathrm{Ind}_{M_i N_i'}^G(\rho_\eta) = \pi_{\eta^+} \oplus \pi_{\eta^-}$. Moreover, since the character of this induced representation vanishes on G_e, we have $\Theta^e_{\pi_{\eta^-}} = -\Theta^e_{\pi_{\eta^+}}$.

For $\kappa \in \hat{R}$, we let $s(\kappa) = |\{1 \leq i \leq r | \kappa(D_i) = -1\}|$. Note, if $s(\kappa) = 0$, then $\kappa = 1$, and the claim is trivially true. Suppose $s \geq 0$ and the claim holds for any $\kappa \in \hat{R}$ with $s(\kappa) = s$. Suppose $s(\kappa) = s+1$. Then we fix some $1 \leq i \leq r$ for which $\kappa(D_i) = -1$. Then consider M_i and R_i as above. Let $\eta = \kappa|_{R_i}$, and suppose ρ_η is the corresponding component of $\mathrm{Ind}_{P_i}^{M_i}(\sigma)$. Then $\kappa = \eta^-$, so by our discussion above, we have $\Theta^e_{\pi_\kappa} = -\Theta^e_{\pi_{\eta^+}}$. Moreover $s(\eta^+) = s$, so, by our hypothesis, $\Theta^e(\pi_{\eta^+}) = \eta^+(C_0)\Theta^e_1$. Now, $\Theta^e_{\pi_\kappa} = -\Theta^e_{\pi_{\eta^+}} = -\eta^+(C_0)\Theta^e_1 = \kappa(C_0)\Theta^1_e$. So the claim holds for all κ with $s(\kappa) = s+1$, and by induction the claim holds for all $\kappa \in \hat{R}$.

Now consider the case where $\mathbf{G} = GSpin_{2n}$ and $c_n\tau \not\simeq \tau$. The proof is essentially the same as above, but we give some details for completeness. Let $\Omega_1(\sigma)$, $\Omega_2(\sigma)$, d_1, d_2, d be as in Theorem 2.7(i). If $d_2 = 0$, then the proof is identical to the one above, so we assume $d_2 > 0$ is even. From Theorem 3.4, we know $d = r$. Then, we again see $\Delta' = \emptyset$, so we easily apply the results of Arthur [2] and Herb [18]. Without loss of generality, we assume $\Omega_1(\sigma) = \{1, \ldots, d_1\}$, and $\Omega_2(\sigma) = \{d_1+1, \ldots, r\}$. Then, $R \simeq \mathbb{Z}_2^{r-1}$, with generators $D_1, \ldots, D_{r-1}$, where $D_i = C_i$, for $1 \leq i \leq d_1$, and $D_i = C_iC_r$ for $d_1 + 1 \leq i \leq r-1$. For each $1 \leq i \leq r-1$, we let $\mathbf{M}_i$ and R_i be defined as in the previous cases. We again let $\eta \leftrightarrow \rho_\eta$ be the correspondence between $\hat{R}_i$ and the components of $\mathrm{Ind}_{MN_i'}^{M_i}(\sigma)$. Then, we again have $\hat{R}(\eta) = \{\eta^+, \eta^-\}$, and so $\Theta^e_{\pi_{\eta^-}} = -\Theta^e_{\pi_{\eta^+}}$. Let $\kappa \in \hat{R}$ and let $s(\kappa) = |\{D_i | \kappa(D_i) = -1\}|$. Then $s(1) = 0$, so the claim holds for the case with $s(\kappa) = 0$. If we assume the result when $s(\kappa) = s$,

then the same argument as above shows it holds when $s(\kappa) = s+1$, and so the claim holds by induction. $\square$

4. Parameters and R-groups for $GSpin$ groups

In this section we discuss the computation of Arthur's R-group associated to a parameter $\varphi : W_G \to {}^LG$, in the case when $G = GSpin_m(F)$. We begin with a lemma which applies to split reductive groups in general.

Lemma 4.1. *Suppose $R_{\psi,\pi} \simeq R(\pi)$, whenever $\psi : W_F' \to {}^LL \hookrightarrow {}^LH$, with $\mathbf{L}$ a maximal proper Levi subgroup of a quasi-split connected group $\mathbf{H}$, and ψ is an elliptic parameter for the L-packet $\Pi_\psi(L)$, containing the square integrable representation π. Let $\mathbf{M}$ be an arbitrary Levi subgroup of $\mathbf{G}$, and $\varphi : W_F' \to {}^LM$ an elliptic parameter for an L–packet $\Pi_\varphi(M)$ containing a square integrable representation σ. Then $R_{\varphi,\sigma} \simeq R(\sigma)$.*

Proof. The proof of this relies on the following result.

Lemma 4.2. *Suppose $\mathbf{M} \subset \mathbf{L}$ are Levi subgroups of $\mathbf{G}$. Suppose $\varphi : W_F' \to {}^LM$ is a parameter. Let $S_\varphi = Z_{\hat{G}}(\varphi)$ and $S_{L,\varphi} = Z_{\hat{L}}(\varphi)$. Then $S_{L,\varphi}^\circ = S_\varphi^\circ \cap \hat{L}$.*

Since S_φ is reductive and $S_{L,\varphi}$ is a reductive (Levi subgroup (e.g., by [7] Lemma 2.1) this is a standard result. $\square$

Now we have $W(\mathbf{G}, \mathbf{A_M}) \simeq W(\hat{G}, A_{\hat{M}})$, with the isomorphism given by $s_\alpha \mapsto s_{\check{\alpha}}$. We let $\mathbf{M}_\alpha$ be the Levi subgroup of $\mathbf{G}$ generated by $\mathbf{M}$ and α. Let $R_\alpha(\sigma)$ be the R-group attached to $i_{M_\alpha,M}(\sigma)$. Considering $\varphi : W_F' \to {}^LM \hookrightarrow {}^LM_\alpha$, we let $S_{\varphi,\alpha} = Z_{\hat{M}_\alpha}(\varphi) = S_\varphi \cap \hat{M}_\alpha$. By Lemma 4.2, $S_{\varphi,\alpha}^\circ = S_\varphi^\circ \cap \hat{M}_\alpha$.

We know from Lemma 2.2 of [7] that $(A_{\hat{M}} \cap S_\varphi)^\circ$ is a maximal torus of S_φ°, so we may take $T_\varphi = (A_{\hat{M}} \cap S_\varphi)^\circ$. Then ${}^LM = Z_{{}^LG}(T_\varphi)$ ([7], Lemma 2.1). Since $T_\varphi \subseteq \hat{M} \subseteq \hat{M}_\alpha$, it follows $T_\varphi \subseteq S_{\varphi,\alpha}$, so T_φ is a maximal torus in $S_{\varphi,\alpha}^\circ$.

Let $W_{\varphi,\alpha} = N_{S_{\varphi,\alpha}}(T_\varphi)/Z_{S_{\varphi,\alpha}}(T_\varphi)$, and $W_{\varphi,\alpha}^\circ = N_{S_{\varphi,\alpha}^\circ}(T_\varphi)/Z_{S_{\varphi,\alpha}^\circ}(T_\varphi)$. Lemma 2.2 of [7] tells us that W_φ (respectively, $W_{\varphi,\alpha}$) can be identified with the subgroup of $W(\hat{G}, A_{\hat{M}})$ (respectively, $W(\hat{M}_\alpha, A_{\hat{M}})$) consisting of the elements that can be represented by elements of S_φ (respectively, $S_{\varphi,\alpha}$). Under these identifications, we have $W_{\varphi,\sigma} \cap \hat{M}_\alpha = W_{\varphi,\alpha,\sigma}$.

Now let $R_{\varphi,\alpha,\sigma} = W_{\varphi,\alpha,\sigma}/W_{\varphi,\alpha,\sigma}^\circ$. The hypothesis implies $R_\alpha(\sigma) \simeq R_{\varphi,\alpha,\sigma}$. Let $\alpha \in \Delta'$. Then $\mu_\alpha(\sigma) = 0$. Thus, $s_\alpha \in W(\sigma)$, and $R_\alpha(\sigma) = 1$. Note $s_\alpha \in W(\mathbf{M}_\alpha, \mathbf{A_M}) \simeq W(\hat{M}_\alpha, A_{\hat{M}})$, so $s_{\check{\alpha}} \in$

$W_{\varphi,\sigma} \cap \hat{M}_\alpha = W_{\varphi,\alpha,\sigma}$. Since $R_{\varphi,\alpha,\sigma} \simeq R_\alpha(\sigma) = 1$, we have $s_{\check{\alpha}} \in W^\circ_{\varphi,\alpha,\sigma}$, as claimed. Conversely, assume $s_{\check{\alpha}} \in W^\circ_{\varphi,\sigma}$. As $s_{\check{\alpha}} \in W_{\varphi,\sigma}$, we have $s_\alpha \in W(\sigma)$. Again, considering $\mathbf{M}_\alpha$, we have $R_{\varphi,\alpha,\sigma} = 1$, so $R_\alpha(\sigma) = 1$, which implies $s_\alpha \in W'$. Therefore, $\alpha \in \Delta'$, as claimed.

□

We now return to the setting where $\mathbf{G} = GSpin_m$. Then $\hat{G} = GSO_{2n}(\mathbb{C})$, if $m = 2n$, and $\hat{G} = GSp_{2n}(\mathbb{C})$, if $m = 2n+1$. Since $\mathbf{G}$ is split, we have ${}^LG = \hat{G} \times W_F$. We consider a parameter $\varphi : W_F \to {}^LG$. Let us describe matrix realizations of $GSO_{2n}(\mathbb{C})$ and $GSp_{2n}(\mathbb{C})$. Let

$$\mu = \begin{cases} 1, & \text{if } \hat{G} = GSO_{2n}(\mathbb{C}), \\ -1, & \text{if } \hat{G} = GSp_{2n}(\mathbb{C}), \end{cases} \qquad \hat{w}_n = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix}, \quad J_{2n} = \begin{pmatrix} 0 & \hat{w}_n \\ \mu\hat{w}_n & 0 \end{pmatrix},$$

and

$$\mathcal{G} = \{g \in GL_{2n}(\mathbb{C}) |\, {}^tgJ_{2n}g = \lambda(g)J_{2n}, \text{ for some } \lambda(g) \in \mathbb{C}^\times\}.$$

If $\mu = -1$, then $\mathcal{G}$ is a connected algebraic group denoted by $GSp_{2n}(\mathbb{C})$. If $\mu = 1$, then $\mathcal{G} = GO_{2n}(\mathbb{C})$ has two connected components. In this case, we can define the similitude norm

$$\nu : GO_{2n}(\mathbb{C}) \to \{\pm 1\}, \quad g \mapsto \lambda(g)^{-n} \det(g).$$

The kernel of this map, denoted by $GSO_{2n}(\mathbb{C})$, is the connected component of $GO_{2n}(\mathbb{C})$.

We let $\hat{M}$ be the Siegel parabolic subgroup of $\hat{G}$, so $\hat{M} \simeq GL_n(\mathbb{C}) \times GL_1(\mathbb{C})$. More precisely, for $g \in GL_n(\mathbb{C})$ we let $\hat{\varepsilon}(g) = \hat{w}_n\, {}^tg^{-1}\hat{w}_n^{-1}$. Then

$$\hat{M} = \left\{ \begin{pmatrix} g & 0 \\ 0 & \lambda\hat{\varepsilon}(g) \end{pmatrix} \middle| g \in GL_n(\mathbb{C}), \lambda \in GL_1(\mathbb{C}) \right\}.$$

Let $\hat{A}_{\hat{M}}$ be the split component of $\hat{M}$, so $\hat{M} = \left\{\mathrm{diag}\left\{aI_n, \lambda a^{-1}I_n\right\}\right\}$. If $\hat{G} = GSO_{2n}$, and n is odd, then $W_{\hat{M}} = \{1\}$. Otherwise, $\hat{W}_{\hat{M}} = W(\hat{G}, \hat{A}_{\hat{M}}) = \{1, \hat{w}\}$, where $\hat{w} : (g, \lambda) \mapsto (\lambda\hat{\varepsilon}(g), \lambda)$, and is represented by $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$.

Thus, we know $\mathbf{M} \simeq GL_n \times GL_1$, $\mathbf{A_M} \simeq GL_1 \times GL_1$, and

$$W(\mathbf{G}, \mathbf{A_M}) = \begin{cases} \{1\} & \text{if } G = GSpin_{2n} \text{ and } n \text{ is odd;} \\ \{1, w\} & \text{otherwise,} \end{cases}$$

where $w : (g, \lambda) \mapsto (\lambda\varepsilon(g), \lambda)$, and ε is the dual involution given by $\hat{\varepsilon}$.

Now let σ be an irreducible unitary supercuspidal representation of M, so $\sigma \simeq \sigma_0 \otimes \psi$, with σ_0 an irreducible unitary supercuspidal representation of $GL_n(F)$ and ψ a unitary character of $F^\times$. So, if $\varphi : W_F \to \hat{M}$ is the corresponding Langlands parameter, then $\varphi = \varphi_0 \times \hat{\psi}$, where φ_0 is the Langlands parameter of σ_0 and $\hat{\psi}$ is the character of $\mathbb{C}^\times$ associated to ψ by local class field theory. Since σ_0 is irreducible and supercuspidal, we know φ_0 is irreducible. We abuse notation to write

$$\varphi(w) = \begin{pmatrix} \varphi_0(w) & 0 \\ 0 & \hat{\psi}(w)\hat{\varepsilon}(\varphi_0(w)) \end{pmatrix}.$$

4.1. **Reducibility and poles of L-functions.** Let $\hat{\mathfrak{n}}$ denote the Lie algebra of the unipotent radical of $\hat{M}$. Let ρ_n denote the standard representation of $GL_n(\mathbb{C})$. The adjoint action r of $\hat{M}$ on $\hat{\mathfrak{n}}$ is given as follows:

$$r = \begin{cases} \wedge^2 \rho_n \otimes \rho_1^{-1}, & \text{if } \hat{G} = GSO_{2n}(\mathbb{C}), \\ \operatorname{Sym}^2 \rho_n \otimes \rho_1^{-1}, & \text{if } \hat{G} = GSp_{2n}(\mathbb{C}). \end{cases}$$

More precisely, let $V = \{X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid {}^tX = -\mu X\}$. Then $(g,\lambda) \in \hat{M}$ acts on $X \in V$ by $(g,\lambda)\cdot X = \lambda^{-1} g X \,{}^t g$.

Suppose $L(s, \wedge^2\varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$. Then $\wedge^2\varphi_0 \otimes \hat{\psi}^{-1}$ contains the trivial representation, so there exists a nonzero $X \in M_n(\mathbb{C})$ such that ${}^tX = -X$ and $(\wedge^2\varphi_0 \otimes \hat{\psi}^{-1})(w)\cdot X = X$, for all $w \in W_F$. We have

$$\hat{\psi}(w)^{-1}\varphi_0(w) X \,{}^t\varphi_0(w) = X, \quad \forall w \in W_F. \tag{4.1}$$

It follows that X is a nonzero intertwining operator between ${}^t\varphi_0^{-1}$ and $\hat{\psi}^{-1} \otimes \varphi_0$. Since φ_0 is irreducible, X is invertible. Observe that this can happen only if n is even (every antisymmetric odd dimensional matrix is singular). In addition, it follows from (4.1) that φ_0 factors through $GSp_n(\mathbb{C})$.

Similarly, if we assume that $L(s, \operatorname{Sym}^2\varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$, we obtain that ${}^t\varphi_0^{-1} \simeq \hat{\psi}^{-1} \otimes \varphi_0$ and φ_0 factors through $GO_n(\mathbb{C})$.

On the other hand, if ${}^t\varphi_0^{-1} \simeq \hat{\psi}^{-1} \otimes \varphi_0$, then (4.1) holds for some $X \in GL_n(\mathbb{C})$. By standard arguments, X is symmetric or antisymmetric. It follows that one of the L-functions $L(s, \wedge^2\varphi_0 \otimes \hat{\psi}^{-1})$ or $L(s, \operatorname{Sym}^2\varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$.

We summarize the above considerations in the following lemma:

Lemma 4.3. *Let $\varphi_0 : W_F \to GL_n(\mathbb{C})$ and $\hat{\psi} : W_F \to GL_1(\mathbb{C})$ be irreducible L-parameters. If $\tilde{\varphi}_0 \simeq \hat{\psi}^{-1} \otimes \varphi_0$, then precisely one of the L-functions $L(s, \wedge^2\varphi_0 \otimes \hat{\psi}^{-1})$ or $L(s, \operatorname{Sym}^2\varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$.*

(1) If n is odd, then $L(s, \wedge^2\varphi_0 \otimes \hat{\psi}^{-1})$ is always holomorphic at $s = 0$ and φ_0 factors through $GO_n(\mathbb{C})$.

(2) If n is even, then $L(s, \wedge^2\varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$ if and only if φ_0 factors through $GSp_n(\mathbb{C})$.

Proposition 4.4. *Let $\mathbf{G} = GSpin_{2n}$, $\mathbf{G}' = GSpin_{2n+1}$, and consider the Siegel Levi subgroup $\mathbf{M} \simeq GL_n \times GL_1$. Let $\sigma \simeq \sigma_0 \otimes \psi$ be an irreducible unitary supercuspidal representation of $M = \mathbf{M}(F)$ with corresponding Langlands parameter $\varphi = \varphi_0 \times \hat{\psi}$. Assume $\tilde{\varphi}_0 \simeq \hat{\psi}^{-1} \otimes \varphi_0$. Let $\pi = \mathrm{Ind}_M^G(\sigma)$ and $\pi' = \mathrm{Ind}_M^{G'}(\sigma)$.*

(1) If n is odd, then π and π' are both irreducible and φ_0 factors through $GO_n(\mathbb{C})$.

(2) If n is even, then π is irreducible if and only if π' is reducible. Moreover, π is irreducible if and only if φ_0 factors through $GSp_n(\mathbb{C})$.

Proof. (1) is clear. For (2), assume n is even and consider $\mathbf{G} = GSpin_{2n}$. Then $\pi = \mathrm{Ind}_M^G(\sigma)$ is irreducible if and only if $L(s, \sigma_0 \otimes \psi, \wedge^2\rho_n \otimes \rho_1^{-1})$ has a pole at $s = 0$ [26, 30]. We know from [17], Theorem 1.4 that

$$L(s, \sigma_0 \otimes \psi, \wedge^2\rho_n \otimes \rho_1^{-1}) = L(s, \wedge^2\varphi_0 \otimes \hat{\psi}^{-1}).$$

The statement follows from Lemma 4.3. Similar arguments work for $\mathbf{G}' = GSpin_{2n+1}$. □

4.2. **Centralizers for the Siegel Parabolic.** We wish to compute $S_\varphi = Z_{\hat{G}}(\mathrm{Im}\,\varphi)$. First, we will compute $Z_{\mathcal{G}}(\mathrm{Im}\,\varphi)$, where $\mathcal{G} = GSp_{2n}(\mathbb{C})$ or $GO_{2n}(\mathbb{C})$. Suppose $X \in \mathcal{G}$ centralizes φ, and write $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $A, B, C, D \in M_n(\mathbb{C})$. Computing directly we have, for all $w \in W_F$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \varphi_0(w) & 0 \\ 0 & \hat{\psi}(w)\hat{\varepsilon}(\varphi_0(w)) \end{pmatrix} = \begin{pmatrix} \varphi_0(w) & 0 \\ 0 & \hat{\psi}(w)\hat{\varepsilon}(\varphi_0(w)) \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

which gives

$$A\varphi_0(w) = \varphi_0(w)A,\ D\hat{\varepsilon}(\varphi_0(w)) = \hat{\varepsilon}(\varphi_0(w))D,\ B\hat{\psi}(w)\hat{\varepsilon}(\varphi_0(w)) = \varphi_0(w)B,$$

and $C\varphi_0(w) = \hat{\psi}(w)\hat{\varepsilon}(\varphi_0(w))C$. The irreducibility of φ_0 tells us A and D are scalars (denoted $a_{11}I_n$ and $a_{22}I_n$, respectively) and also shows $C = B = 0$, unless $\varphi_0 \simeq (\hat{\varepsilon} \circ \varphi_0) \otimes \hat{\psi}$. Thus, if $\sigma_0 \not\simeq \tilde{\sigma}_0 \otimes \psi \circ \det$, then $Z_{\mathcal{G}}(\varphi) = \left\{ \begin{pmatrix} aI_n & \\ & \lambda a^{-1} I_n \end{pmatrix} \right\} = \hat{A}_{\hat{M}} \simeq \mathbb{C}^\times \times \mathbb{C}^\times$, and clearly, $Z_{\hat{G}}(\varphi) = Z_{\mathcal{G}}(\varphi)$. So, suppose

$\sigma_0 \simeq \tilde{\sigma}_0 \otimes \psi \circ \det$. Fix an equivalence, B between these two representations, i.e., take B so that $B^{-1}\varphi_0(w)B = \hat{\psi}(w)\hat{\varepsilon}(\varphi_0(w))$. By Schur's Lemma, B is unique up to scalar. We note

$$(B\hat{\varepsilon}(B))^{-1}\varphi_0(w)(B\hat{\varepsilon}(B)) = \hat{\varepsilon}(B)^{-1}(\hat{\psi}(w)\hat{\varepsilon}(\varphi_0)(w))\hat{\varepsilon}(B) = \hat{\varepsilon}(B^{-1}\varphi_0(w)B)\hat{\psi}(w) = \varphi_0(w),$$

and thus $B\hat{\varepsilon}(B) = cI_n$, for some $c \in \mathbb{C}^\times$. We write this as $B\hat{w}_n = c\hat{w}_n\,{}^tB$. Note that if $J = B\hat{w}_n$, then we have ${}^tJ = c^{-1}J$, so $c = \pm 1$, and J is a symmetric or symplectic form fixed by φ_0 up to the multiplier $\hat{\psi}$.

Now, we have $X = \begin{pmatrix} a_{11}I_n & a_{12}B \\ a_{21}B^{-1} & a_{22}I_n \end{pmatrix}$, and since $X \in \mathcal{G}$, we have

$${}^tX \begin{pmatrix} & \hat{w}_n \\ \mu\hat{w}_n & \end{pmatrix} X = \begin{pmatrix} & \lambda\hat{w}_n \\ \lambda\mu\hat{w}_n & \end{pmatrix}$$

or,

$$\begin{pmatrix} a_{11}a_{21}(1+\mu c)\hat{w}_n B^{-1} & (a_{11}a_{22} + a_{21}a_{12}\mu c)\hat{w}_n \\ (a_{11}a_{22} + a_{12}a_{21}\mu c)\mu\hat{w}_n & a_{12}a_{22}(1+\mu c)\,{}^tB\hat{w}_n \end{pmatrix} = \begin{pmatrix} & \lambda\hat{w}_n \\ \lambda\mu\hat{w}_n & \end{pmatrix}.$$

We see this is equivalent to the 2×2 complex matrix $Y = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ satisfying ${}^tY \begin{pmatrix} & 1 \\ \mu c & \end{pmatrix} Y = \begin{pmatrix} & \lambda \\ \lambda\mu c & \end{pmatrix}$. Thus $X \mapsto Y$ is an isomorphism,

$$Z_{\mathcal{G}}(\varphi) \simeq \begin{cases} GSp_2(\mathbb{C}) \simeq GL_2(\mathbb{C}) & \text{if } \mu c = -1; \\ GO_{1,1}(\mathbb{C}) & \text{if } \mu c = 1. \end{cases} \tag{4.2}$$

This is equal to S_φ if $\hat{G} = GSp_{2n}(\mathbb{C})$.

Now, let $\hat{G} = GSO_{2n}(\mathbb{C})$, so $\mu = 1$. Let $X = \begin{pmatrix} a_{11}I_n & a_{12}B \\ a_{21}B^{-1} & a_{22}I_n \end{pmatrix} \in Z_{\mathcal{G}}(\varphi)$. We have to determine whether $X \in \hat{G}$. Assume first $c = -1$. Then

$$\begin{pmatrix} & (a_{11}a_{22} - a_{12}a_{21})\hat{w}_n \\ (a_{11}a_{22} - a_{12}a_{21})\hat{w}_n & \end{pmatrix} = \begin{pmatrix} & \lambda\hat{w}_n \\ \lambda\hat{w}_n & \end{pmatrix},$$

so $\lambda = a_{11}a_{22} - a_{12}a_{21}$. We use the formula $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - CA^{-1}B)$, if A is invertible. Therefore, if $a_{11} \neq 0$, we have

$$\det X = a_{11}^n \det(a_{22}I_n - a_{21}a_{11}^{-1}a_{12}B^{-1}B) = \det(a_{11}a_{22} - a_{12}a_{21})I_n = \lambda^n.$$

The similitude norm $\nu(X) = \lambda^{-n} \det X = 1$, so $X \in GSO_{2n}(\mathbb{C})$. If $a_{11} = 0$, then

$$\det X = \det \begin{pmatrix} 0 & a_{12}B \\ a_{21}B^{-1} & a_{22}I_n \end{pmatrix} = (-1)^n \det \begin{pmatrix} a_{21}B^{-1} & a_{22}I_n \\ 0 & a_{12}B \end{pmatrix} = \lambda^n,$$

and again $X \in GSO_{2n}(\mathbb{C})$.

Assume $c = 1$. Then we have

$$\begin{pmatrix} 2a_{11}a_{21}\hat{w}_n B^{-1} & (a_{21}a_{12} + a_{11}a_{22})\hat{w}_n \\ (a_{11}a_{22} + a_{12}a_{21})\hat{w}_n & 2a_{12}a_{22}\, {}^tB\hat{w}_n \end{pmatrix} = \begin{pmatrix} & \lambda\hat{w}_n \\ \lambda\hat{w}_n & \end{pmatrix}.$$

It follows $a_{12} = a_{21} = 0$ or $a_{11} = a_{22} = 0$. If $a_{12} = a_{21} = 0$, then $a_{22} = \lambda a_{11}^{-1}$ and $X = \begin{pmatrix} a_{11}I_n & \\ & \lambda a_{11}^{-1}I_n \end{pmatrix}$. The similitude norm $\nu(X) = \lambda^{-n}\det(X) = 1$, so $X \in GSO_{2n}(\mathbb{C})$. If $a_{11} = a_{22} = 0$, then $X = \begin{pmatrix} & a_{12}B \\ \lambda a_{12}^{-1}B^{-1} & \end{pmatrix}$ and

$$\nu(X) = \lambda^{-n}\det(X) = (-1)^n\lambda^{-n}\lambda^n = (-1)^n.$$

It follows that $X \in GSO_{2n}(\mathbb{C})$ if n is even and $X \notin GSO_{2n}(\mathbb{C})$ if n is odd. Therefore,

$$S_\varphi = Z_{\hat{G}}(\varphi) \simeq \begin{cases} GSp_2(\mathbb{C}) \simeq GL_2(\mathbb{C}) & \text{if } c = -1; \\ GO_{1,1}(\mathbb{C}) & \text{if } c = 1, \text{n even}; \\ \mathbb{C}^\times & \text{if } c = 1, \text{n odd}. \end{cases}$$

4.3. **The Arthur R-group.** Now we can compute R_φ, the Arthur R-group of φ. We summarize the above computation as follows.

Theorem 4.5. *Let $\mathbf{G} = GSpin_{2n+1}$ or $GSpin_{2n}$ and consider the Siegel Levi subgroup $\mathbf{M} \simeq GL_n \times GL_1$. Let $\sigma \simeq \sigma_0 \otimes \psi$ be an irreducible unitary supercuspidal representation of $M = \mathbf{M}(F)$ with corresponding Langlands parameter $\varphi = \varphi_0 \otimes \hat{\psi}$.*

(1) If $\varphi_0 \not\simeq \tilde{\varphi}_0 \otimes \hat{\psi}$, then $R_{\varphi,\sigma} = R_\varphi = 1$.

(2) Assume $\varphi_0 \simeq \tilde{\varphi}_0 \otimes \hat{\psi}$. If $\mathbf{G} = GSpin_{2n+1}$, then

$$R_{\varphi,\sigma} = R_{\varphi} = \begin{cases} 1, & \text{if } \varphi_0 \text{ factors through } GO_n(\mathbb{C}); \\ \mathbb{Z}_2, & \text{if } \varphi_0 \text{ factors through } GSp_n(\mathbb{C}). \end{cases}$$

If $\mathbf{G} = GSpin_{2n}$, then

$$R_{\varphi,\sigma} = R_{\varphi} = \begin{cases} 1, & \text{if } \varphi_0 \text{ factors through } GSp_n(\mathbb{C}); \\ \mathbb{Z}_2, & \text{if } \varphi_0 \text{ factors through } GO_n(\mathbb{C}) \text{ and } n \text{ is even,} \\ 1, & \text{if } \varphi_0 \text{ factors through } GO_n(\mathbb{C}) \text{ and } n \text{ is odd.} \end{cases}$$

Corollary 4.6. *For $\mathbf{G} = GSpin_{2n+1}$ or $GSpin_{2n}$, and $\mathbf{M} \simeq GL_n \times GL_1$, we have $R(\sigma) \simeq R_{\varphi,\sigma}$, as conjectured by Arthur.*

Proof. This follows from the theorem and Proposition 4.4. □

4.4. **Centralizers (The General Case).** Let V be a finite dimensional complex vector space. Let B be a non-degenerate bilinear form on V and

$$\mathcal{G}_B = \{g \in GL_n(V) \mid B(gu, gv) = \lambda(g)B(u, v), \text{ for some } \lambda(g) \in \mathbb{C}^\times, \forall u, v \in V\}.$$

Lemma 4.7. *Let $\varphi : W_F' \to GL_n(V)$ be an irreducible parameter and let B be a non-degenerate bilinear form on V. Then φ factors through $\mathcal{G}_B$ if and only if $\varphi \simeq \chi \otimes {}^t\varphi^{-1}$, where $\chi = \lambda \circ \varphi$. If φ factors through $\mathcal{G}_B$, then B is unique up to a scalar multiple.*

Proof. Suppose that φ factors through $\mathcal{G}_B$. Let A be the matrix corresponding to B, $B(u, v) = {}^tuAv$. Then for all $w \in W_F'$, ${}^t\varphi(w)A\varphi(w) = \lambda(\varphi(w))A$. It follows

$$\varphi(w) = \chi(w)A\, {}^t\varphi(w)^{-1}A^{-1}, \quad \forall w \in W_F', \tag{4.3}$$

where $\chi = \lambda \circ \varphi$. Hence, $\varphi \simeq \chi \otimes {}^t\varphi^{-1}$. If B' is another non-degenerate bilinear form on V such that φ factors through $\mathcal{G}_{B'}$, and if A' is the corresponding matrix, we have

$$\varphi(w) = \chi(w)A'\, {}^t\varphi(w)^{-1}(A')^{-1}, \quad \forall w \in W_F'. \tag{4.4}$$

By transposing and taking inverses, equation (4.3) gives us ${}^t\varphi(w)^{-1} = \chi(w)^{-1}A^{-1}\varphi(w)A$. We substitute this in equation (4.4) and we obtain

$$\varphi(w) = A'A^{-1}\varphi(w)A(A')^{-1}, \quad \forall w \in W_F'.$$

Since φ is irreducible, it follows $A'A^{-1} = cI$ and $A' = cA$.

Next, suppose $\varphi \simeq \chi \otimes {}^t\varphi^{-1}$ for a character χ. Let A be a matrix such that

$$\varphi(w) = \chi(w) A \, {}^t\varphi(w)^{-1} A^{-1},$$

for all $w \in W_F'$. Standard arguments show that $A \, {}^tA^{-1} = cI$ and $c = \pm 1$. It follows that $B(u,v) = {}^tuAv$ is a non-degenerate bilinear form such that φ factors through $\mathcal{G}_B$. □

Lemma 4.8. *Let $\varphi : W_F' \to \mathcal{G}_B$ be a parameter. Suppose $\varphi = \underbrace{\varphi_0 \oplus \cdots \oplus \varphi_0}_{m-summands}$, where φ_0 is an irreducible parameter such that φ_0 factors through $\mathcal{G}_{B_0}$ for some non-degenerate bilinear form B_0. Then*

$$Z_{\mathcal{G}_B}(\operatorname{Im}\varphi) \simeq \begin{cases} GO(m,\mathbb{C}), & if\ B\ and\ B_0\ are\ both\ symmetric\ or\ both\ symplectic, \\ GSp(m,\mathbb{C}), & otherwise. \end{cases}$$

Proof. Let V_0 denote the space of the representation φ_0. Then $V \simeq W \otimes V_0$, where $W = \operatorname{Hom}_{W_F'}(V_0, V)$ with trivial W_F'-action. The map $W \otimes V_0 \to V$ is given by

$$f \otimes v \mapsto f(v), \quad f \in W, v \in V_0. \tag{4.5}$$

For $f, g \in W$, we define a bilinear form $B_{f,g}$ on V_0 by $B_{f,g}(u,v) = B(f(u), g(v))$. Then

$$\begin{aligned} B_{f,g}(\varphi_0(w)u, \varphi_0(w)v) &= B(f(\varphi_0(w)u), g(\varphi_0(w)v)) \\ &= B(\varphi(w)f(u), \varphi(w)g(v)) \\ &= \lambda \circ \varphi(w) B_{f,g}(u,v). \end{aligned}$$

It follows from Lemma 4.7 that $B_{f,g}$ is a scalar multiple of B_0; denote that scalar by $\langle f, g \rangle$. The map $(f,g) \mapsto \langle f, g \rangle$ defines a bilinear form $\langle \, , \rangle$ on W. The form $\langle \, , \rangle$ is symmetric if B and B_0 are both symmetric or both symplectic, and symplectic otherwise. Moreover, if we identify $W \otimes V_0$ and V using equation (4.5), we have

$$B(f \otimes u, g \otimes v) = B(f(u), g(v)) = B_{f,g}(u,v) = \langle f, g \rangle B_0(u,v),$$

for all $f, g \in W$, $u, v \in V_0$.

Now, $\operatorname{Im}\varphi = \{I_W \otimes g \mid g \in \operatorname{Im}\varphi_0\}$ and

$$Z_{GL(V)}(\operatorname{Im}\varphi) = \{g \otimes z \mid g \in GL(W), z = cI_{V_0}, c \in \mathbb{C}^\times\} = \{g \otimes I_{V_0} \mid g \in GL(W)\}.$$

The element $g \otimes I_{V_0}$ belongs to $\mathcal{G}_B$ if for some $\lambda \in \mathbb{C}^\times$,

$$B((g \otimes I_{V_0})(f \otimes u), (g \otimes I_{V_0})(h \otimes v)) = \lambda B(f \otimes u, h \otimes v), \quad \forall f, h \in W, \forall u, v \in V_0,$$

that is,

$$\langle gf, gh\rangle = \lambda\langle f, h\rangle, \quad \forall f, h \in W.$$

It follows $Z_{\mathcal{G}_B}(\operatorname{Im}\varphi) \simeq \mathcal{G}_{\langle\,,\rangle}$. $\square$

4.5. **Reducibility for generic representations.** Let $G = GSpin_m(F)$ and let $P = MN$ be a maximal Levi subgroup. Then $M \simeq GL_k(F) \times GSpin_\ell(F)$, where $2k + \ell = m$. In the case $\ell = 0$ or 1, P is the Siegel parabolic subgroup and that case was considered earlier. We assume $\ell > 2$. Let $\pi = \sigma \otimes \tau$ be an irreducible unitary generic supercuspidal representation of M. Let $\alpha \in \Delta$ be the unique reduced root of Θ in $\mathbf{N}$ and set $\tilde{\alpha} = \langle\rho, \alpha\rangle^{-1}\alpha$, where ρ is half the sum of positive roots in $\mathbf{N}$. We have $\tilde{\alpha}/i \otimes \pi = \nu^{1/i}\sigma \otimes \tau$. Assume $\sigma \simeq \tilde{\sigma} \otimes \omega_\tau$. According to [26], exactly one of the following representations is reducible: $\operatorname{Ind}_P^G(\sigma \otimes \tau)$, $\operatorname{Ind}_P^G(\nu^{1/2}\sigma \otimes \tau)$, or $\operatorname{Ind}_P^G(\nu\sigma \otimes \tau)$.

Lemma 4.9. *Let $G = GSpin_m(F)$ and $M \simeq GL_k(F) \times GSpin_\ell(F)$, where $2k + \ell = m$, $\ell > 2$. Let $\pi = \sigma \otimes \tau$ be an irreducible unitary generic supercuspidal representation of M. Assume $\sigma \simeq \tilde{\sigma} \otimes \omega_\tau$. Let φ_0 denote the Langlands parameter of σ.*

(1) Suppose $G = GSpin_{2n+1}(F)$. If $\operatorname{Ind}_P^G(\nu^{1/2}\sigma \otimes \tau)$ reduces, then φ_0 factors through $GO_k(\mathbb{C})$. Otherwise, φ_0 factors through $GSp_k(\mathbb{C})$.

(2) Suppose $G = GSpin_{2n}(F)$. If $\operatorname{Ind}_P^G(\nu^{1/2}\sigma \otimes \tau)$ reduces, then φ_0 factors through $GSp_k(\mathbb{C})$. Otherwise, φ_0 factors through $GO_k(\mathbb{C})$.

Proof. Let $\hat{\mathfrak{n}}$ denote the Lie algebra of the unipotent radical of $\hat{M}$. Denote the standard representations of the groups $GL_k(\mathbb{C})$, $GSp_{2\ell}(\mathbb{C})$ and $GSO_{2\ell}(\mathbb{C})$ by ρ_k, $R^1_{2\ell}$ and $R^2_{2\ell}$, respectively. Let μ be the similitude character of $GSp_{2\ell}(\mathbb{C})$ or $GSO_{2\ell}(\mathbb{C})$. The adjoint action r of $\hat{M}$ on $\hat{\mathfrak{n}}$ is described in Proposition 5.6 of [4]. In particular, we have

(a) If $G = GSpin_{2n+1}(F)$, then $r = r_1 \oplus r_2$, where

$$r_1 = \rho_k \otimes \widetilde{R^1_{\ell-1}}, \quad r_2 = \operatorname{Sym}^2 \rho_k \otimes \mu^{-1}.$$

(b) If $G = GSpin_{2n}(F)$, then $r = r_1 \oplus r_2$, where

$$r_1 = \rho_k \otimes \widetilde{R^2_\ell}, \quad r_2 = \wedge^2 \rho_k \otimes \mu^{-1}.$$

Let $P_{\pi,1}$ and $P_{\pi,2}$ be the polynomials defined in [26]. The Langlands-Shahidi L-function attached to π and r_i is defined as

$$L(s, \pi, r_i) = P_{\pi,i}(q^{-s})^{-1}.$$

Assume $G = GSpin_{2n}(F)$. Theorem 8.1 of [26] tells us that $\mathrm{Ind}_P^G(\nu^{1/2}\sigma \otimes \tau)$ is reducible if and only if $P_{\pi,2}(1) = 0$. Equivalently, $L(s, \pi, r_2)$ has a pole at $s = 0$. In order to complete the proof, we need the following result.

Lemma 4.10. *Let* $\mathbf{G} = GSpin_m$ *and* $\mathbf{M} \simeq GL_k \times GSpin_\ell$. *Let* $\pi = \sigma \otimes \tau$ *be an irreducible admissible generic representation of* M. *Let* $\varphi = (\varphi_0, \varphi_\tau)$ *be the Langlands parameter attached to* π.

a) If $m = 2n$ *is even, then*

$$\begin{aligned} L(s, \pi, r_2) &= L(s, \sigma \otimes \tau, \wedge^2 \rho_k \otimes \mu^{-1}) \\ &= L(s, \sigma \otimes \psi, \wedge^2 \rho_k \otimes \rho_1^{-1}) = L(s, \wedge^2 \varphi_0 \otimes \hat{\psi}^{-1}). \end{aligned} \tag{4.6}$$

b) If $m = 2n + 1$ *is odd, then*

$$\begin{aligned} L(s, \pi, r_2) &= L(s, \sigma \otimes \tau, \mathrm{Sym}^2 \rho_k \otimes \mu^{-1}) \\ &= L(s, \sigma \otimes \psi, \mathrm{Sym}^2 \rho_k \otimes \rho_1^{-1}) = L(s, \mathrm{Sym}^2 \varphi_0 \otimes \hat{\psi}^{-1}). \end{aligned} \tag{4.7}$$

Proof. We continue with the notation of the proof of Lemma 4.9. Suppose $m = 2n$. Then $\varphi_\tau : W_F \to GSO_{2\ell}(\mathbb{C})$. First, we prove (4.6) holds for any unramified generic π. By Prop. 2.3(a) of [5] we know $Z(GSp_{2\ell}(F))^\circ = \{e_0^*(\lambda) | \lambda \in F^\times\}$. So, the central character of τ is given by

$$\omega_\tau(\lambda)\, \mathrm{Id}_{V_\tau} = \tau(e_0^*(\lambda)).$$

Let $\hat{\psi} : W_F \to \mathbb{C}^\times$ be the character attached to ω_τ by Class Field Theory. In particular, $\omega_\tau(\varpi_F) = \hat{\psi}(\mathrm{Fr}_F)$, where Fr_F is the Frobenius class of F. Let $\hat{T}$ be the maximal torus of $GSO_{2m}(\mathbb{C})$. Then $\mu(t) = e_0^*(t)$, (by [5] pg. 149). Now, we have

$$L(s, \pi, r_2) = L(s, \sigma \otimes \tau, \wedge^2 \rho_k \otimes \mu^{-1}) = L(s, \wedge^2 \rho_k \otimes \mu^{-1}(\varphi_0, \varphi_\tau)).$$

Note, for $w \in W_F$, we have $\wedge^2 \rho_k \otimes \mu^{-1}(\varphi_0, \varphi_\tau)(w) = \wedge^2(\varphi_0(w))\mu^{-1}(\varphi_\tau(w))$. Now

$$\mu^{-1}(\varphi_\tau(\mathrm{Fr}_F)) = (e_0^*(\varphi_\tau(\mathrm{Fr}_F)))^{-1} = \tau(e_0^*(\varpi_F))^{-1} = \omega_\tau(\varpi_F)^{-1}.$$

So

$$L(s, \pi, r_2) = L(s, \wedge^2 \rho_k \varphi_0 \otimes \hat{\psi}^{-1}) = L(s, \sigma \otimes \omega_\tau, \wedge^2 \rho_k \otimes \rho_1^{-1}).$$

If S_n denotes the n-dimensional complex representation of $SL(2, \mathbb{C})$, then $\mathrm{Im}(S_n)$ is orthogonal or symplectic. Therefore, $\mu(\varphi \otimes S_n) = \mu(\varphi)$. We conclude that equation (4.6) holds if π has an Iwahori fixed vector. In addition, for φ unramified, the Artin ε–factor associated to $\mu(\varphi \otimes S_n)$ is equal to 1.

Now, we apply Theorem 3.5 of [26] to $\pi = \sigma \otimes \tau$ and independently we apply the same theorem to $\sigma \otimes \omega_\tau$. The theorem guarantees existence of the γ–factors $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w})$ and $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$,

with the subscripts determined by the ordering of the components of the adjoint representations of the L–groups of the Levi subgroups in two distinct situations. Moreover, conditions 1, 3, and 4 from the theorem determine these γ–factors uniquely. These conditions are satisfied by $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w})$ and independently by $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$. In the inductive property 3 for $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$, only σ can be induced from a smaller parabolic subgroup, not ω_τ. Therefore, if we look at the inductive property for $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$, the same conditions are satisfied for $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w})$. Even though we can have additional conditions for $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w})$, the conditions for $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$ are enough to guarantee uniqueness. Since we have equality of γ–factors for representations with Iwahori fixed vectors, we conclude that $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w}) = \gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$. The definition of L-functions from [26] then implies (4.6).

The proof of the case $\mathbf{G} = GSpin_{2n+1}$ is similar. $\square$

We return to the proof of Lemma 4.9. It follows from Lemma 4.3 and Lemma 4.10 that $\mathrm{Ind}_P^G(\nu^{1/2}\sigma \otimes \tau)$ is reducible if and only if φ_0 factors through $GSp_k(\mathbb{C})$. Finally, we remark the claim will follow in general from the generic L–packet conjecture of Shahidi [26].

The proof for $G = GSpin_{2n+1}(F)$ is similar. $\square$

Let $G = GSpin_{2\ell+1}(F)$ and let τ be a generic discrete series representation of G. As in [23], let $\mathrm{Jord}(\tau)$ denote the set of pairs (ρ, a), where $\rho \in {}^0\mathcal{E}(GL(d_\rho, F))$ and $a \in \mathbb{Z}^+$ such that $\delta(\rho, a) \rtimes \tau$ is irreducible and there exists an integer a' of the same parity as a such that $\delta(\rho, a') \rtimes \tau$ is reducible. Here $\delta \rtimes \tau = \mathrm{Ind}_{GL_d(F) \times G(\ell)}^{G(\ell+d)}(\delta \otimes \tau)$. The L-parameter of τ is given by

$$\varphi_\tau = \bigoplus_{(\rho,a) \in \mathrm{Jord}(\tau)} \varphi_\rho \otimes S_a,$$

where φ_ρ is the L-parameter of ρ.

Theorem 4.11. *Let $\mathbf{G} = GSpin_{2n+1}$ and consider the Levi subgroup $M \simeq GL_k(F) \times GSpin_{2\ell+1}(F)$. Let $\pi = \sigma \otimes \tau$ be a generic discrete series representation of M. Let φ be the L-parameter of π. Then $R_{\varphi,\pi} \simeq R(\pi)$.*

Proof. The parameter φ can be written as $\varphi \simeq \varphi_\sigma \oplus \varphi_\tau \oplus (\hat{\varepsilon}(\varphi_\sigma) \otimes \hat{\psi})$, where $\hat{\psi}$ is the character corresponding to the central character of τ, (restricted to the connected component of the center) by Class Field Theory. The representation σ is of the form $\sigma \simeq \delta(\rho, a)$, where $\rho \in {}^0\mathcal{E}(GL(d, F))$ and $a \in \mathbb{Z}^+$, $da = k$. Then $\varphi_\sigma = \varphi_\rho \otimes S_a$.

If $\sigma \not\simeq \tilde{\sigma} \otimes \omega_\tau$, it is easy to show $R_{\varphi,\pi} = 1$ and $R(\pi) = 1$. Assume $\sigma \simeq \tilde{\sigma} \otimes \omega_\tau$. Then $\hat{\varepsilon}(\varphi_\sigma) \otimes \hat{\psi} \simeq \varphi_\sigma$, so $\varphi \simeq \varphi_\sigma \oplus \varphi_\tau \oplus \varphi_\sigma$.

If $(\rho, a) \in \mathrm{Jord}(\tau)$, then the multiplicity of φ_σ in $\varphi \simeq \varphi_\sigma \oplus \varphi_\tau \oplus \varphi_\sigma$ is three. Lemma 4.8 implies $R_\varphi = 1$. On the other hand, since $(\rho, a) \in \mathrm{Jord}(\tau)$, we have $\sigma \rtimes \tau$ is irreducible, so $R(\pi) = 1$.

Now, consider the case $\sigma \simeq \tilde{\sigma} \otimes \omega_\tau$ and $(\rho, a) \notin \mathrm{Jord}(\tau)$. There exist a supercuspidal generic representation τ_{cusp} of $GSpin_{2m+1}(F)$ and an irreducible generic representation θ of $GL_r(F)$ such that τ is a subrepresentation of

$$\theta \rtimes \tau_{cusp} = i_{GSpin_{2\ell+1}(F), GL_r(F) \times GSpin_{2m+1}(F)}(\theta \otimes \tau_{cusp}).$$

We apply the Langlands classification for $GL_r(F)$ in the subrepresentation setting. It follows that there exist $\delta(\rho_1, a_1), \delta(\rho_2, a_2), \ldots, \delta(\rho_s, a_s)$ and real numbers $b_1 < b_2 < \cdots < b_s$ such that θ is the unique subrepresentation of the induced representation

$$\nu^{b_1}\delta(\rho_1, a_1) \times \nu^{b_2}\delta(\rho_2, a_2) \times \cdots \times \nu^{b_s}\delta(\rho_s, a_s).$$

For $i \in \{1, \ldots, s\}$, define $[i] = \{j \in \{1, \ldots, s\} \mid \rho_i \simeq \rho_j\}$. The Casselman square integrability criterion for τ implies that for $i = 1, \ldots, s$, there exists $j \in [i]$ such that the representation

$$\nu^{b_j}\delta(\rho_j, a_j) \rtimes \tau_{cusp}$$

is reducible, and $b_i - b_j \in \mathbb{Z}$. This implies $b_j \in \frac{1}{2}\mathbb{Z}$ and therefore $b_i \in \frac{1}{2}\mathbb{Z}$.

Assume first $\sigma \rtimes \tau$ is reducible. Then $R(\pi) \simeq \mathbb{Z}_2$. It can be shown, taking into account the structure of θ, that reducibility of $\sigma \rtimes \tau$ implies reducibility of $\sigma \rtimes \tau_{cusp}$. Then there exists $b \geq 0$, $b \in \{-\frac{(a-1)}{2}, -\frac{(a-1)}{2} + 1, \ldots, \frac{(a-1)}{2}\}$ such that $\nu^b\rho \rtimes \tau_{cusp}$ is reducible. Since τ_{cusp} is supercuspidal and generic, we have $b = 0, 1/2$ or 1. If $b = 1/2$, then a is even. In addition, Lemma 4.9 implies that φ_ρ factors through $GO_d(\mathbb{C})$. Then $\varphi_\sigma = \varphi_\rho \otimes S_a$ factors through $GSp_k(\mathbb{C})$. Now Lemma 4.8 tells us that $S_\varphi \simeq GO(2, \mathbb{C})$. It follows $R_\varphi = R_{\varphi,\pi} \simeq \mathbb{Z}_2$. If $b = 0$ or 1, then a is odd. In addition, Lemma 4.9 implies that φ_ρ factors through $GSp_d(\mathbb{C})$. Then $\varphi_\sigma = \varphi_\rho \otimes S_a$ factors through $GSp_k(\mathbb{C})$. As before, we obtain $R_{\varphi,\pi} \simeq \mathbb{Z}_2$.

It remains to consider the case when $\sigma \rtimes \tau$ is irreducible, $\sigma \simeq \tilde{\sigma} \otimes \omega_\tau$ and $(\rho, a) \notin \mathrm{Jord}(\tau)$. Irreducibility of $\sigma \rtimes \tau$ implies $R(\pi) = 1$. Let $b \in \{0, 1/2, 1\}$ such that $\nu^b\rho \rtimes \tau_{cusp}$ is reducible. Since $(\rho, a) \notin \mathrm{Jord}(\tau)$, a and $2b + 1$ are not of the same parity. Therefore, if $b = 1/2$, then a is odd. Then φ_ρ factors through $GO_d(\mathbb{C})$ and $\varphi_\sigma = \varphi_\rho \otimes S_a$ factors through $GO_k(\mathbb{C})$. It follows $R_\varphi = R_{\varphi,\pi} = 1$. Similarly, if $b = 0$ or 1, then a is even, φ_ρ factors through $GSp_d(\mathbb{C})$ and $\varphi_\sigma = \varphi_\rho \otimes S_a$ factors through $GO_k(\mathbb{C})$, implying $R_\varphi = R_{\varphi,\pi} = 1$. □

Theorem 4.12. *Let* $\mathbf{G} = GSpin_{2n+1}$ *and* $\mathbf{P} = \mathbf{MN}$ *be an arbitrary parabolic subgroup of* $\mathbf{G}$. *Suppose* π *is a discrete series representation of* M *and* $\varphi = \varphi_\pi : W_F \to {}^L M$ *is the corresponding*

Langlands parameter for the L–packet $\Pi_M(\varphi)$ containing π. Let $R(\pi)$ be the Knapp-Stein R–group of π and $R_{\varphi,\pi}$ the Arthur R–group attached to φ and π. Then $R(\pi) \simeq R_{\varphi,\pi}$, and this isomorphism is realized by the map $\alpha \mapsto \check{\alpha}$ between roots and coroots.

Proof. By Lemma 4.1 it is enough to prove this isomorphism in the case $\mathbf{P}$ is maximal. This, however, is exactly the content of Corollary 4.6 and Theorem 4.11. □

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